The Effects of Information on Abandonment and Congestion in Non-Stationary Priority Queues

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In observable priority service systems, e.g., the emergency department of a hospital, customers are typically left to infer their own information about their queue positions in the waiting room. Because such inferences are likely inaccurate, the system manager may choose to provide accurate information, e.g., through Smartphone Apps. In this paper, we examine the impact of providing (queue position) information on key operational performance measures of the system. We consider a two-class, multi-server priority queue with time-varying arrivals and state-dependent abandonment rates, and compare two levels of information: no information (where abandonment rates are driven by customers' class-dependent perceptions of their queue positions) and full information (where abandonment rates are driven by customers' true queue positions). For each level, we derive time-varying fluid approximations of the queue length process, and establish the existence, uniqueness, and asymptotic stability of a periodic equilibrium for the fluid models. The approximations allow us to analytically compare the two information levels based on class-dependent performance measures. The results yield new insights on (1) the interactions between information system features of priority service and time-varying arrivals; (2) trade-offs in effects of information between priority classes and different performance metrics; and (3) the impact of customers' perception and system load on the effects of information. Specifically, we find that providing accurate information may have opposing effects on the abandonment rates of different priority classes. These effects depend on the system load, and hence in certain cases be controlled (e.g., through staffing decision) to eliminate the trade-offs.

Key words: State-dependent abandonment; information-dependent abandonment; priority queues; time-varying arrivals; service systems

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1. Introduction

Customer abandonment is prevalent in service systems and can have a significant impact on system performance. In most service settings, some form of information on the current "state" of the system, or waiting time estimates, is available. This information may be communicated (and, hence, fully controlled) by the provider in unobservable (virtual) queues. In observable service settings, customers are left to infer their own information. Because such inferences are noisy and possibly inaccurate, the system manager may choose to provide accurate information through technologyenabled solutions, e.g., Smartphone Apps. In this paper, we examine the impact of providing (queue position) information on key operational performance measures of the system.

Our primary motivation comes from the Emergency Department (ED) of a hospital, where patients with different medical urgency (and, hence, service priority) share a common waiting room and typically do not receive real-time information regarding their anticipated delays, position, or priority level. Previous (observational) empirical studies (e.g., Batt and Terwiesch 2015, Bolandifar et al. 2019) have found that observable (time-varying) characteristics of the waiting room (number waiting, service speed) and those of the patients (severity levels) impact their abandonment decisions. This raises the question of whether abandonment rate and other system-level performance measures of EDs can be improved by providing information, in form of queue positions or waiting time predictions, to the patients. Examining the effects of providing information about queue positions or delays through experimental or quasi-experimental studies is particularly challenging in the complex setting of an ED. In addition to ethical and quality of care concerns, abandonment rates are also typically small, hence requiring long experiments to obtain sufficiently large samples. A rare example is Westphal et al. (2022) which evaluates the impact of providing operational (current and next stage of care) and wait time information (expected remaining time in ED) through a field-study implementation of a web-based phone application called MyED (Westphal et al. 2020).

In this paper, we propose a modeling framework to evaluate the impact of providing delay information in observable service settings, such as the ED. Our primary goal is to gain insights into the key trade-offs that impact the value of information, and the interaction with key system features of priority service and time-varying arrivals, which are prevalent in the ED and other observable service settings.

We consider a Markovian multi-server two-class queueing system with time-varying Poisson arrivals, operating under a strict priority discipline. We model abandonment rates as state-dependent through an increasing and concave function that maps customers' *perceived* queue positions to their abandonment rates. The perceived queue positions depend on the information available to the customers. Specifically, our analysis compares the following two levels of information. (1) *No information*: customers only observe the total queue length, but neither the priority classes nor the positions of individual customers in the queue. In this case, we assume that customers perceive their position to be proportional to the total queue length, where this proportion could differ between priority classes and could depend on elapsed waiting times. (2) *Full information*: customers are aware of the priority classes and customer positions, i.e., observe their true queue position.

In the ED setting, the high- and low-priority classes in our model correspond to emergency severity index (ESI) levels 2 and 3, and ESI levels 4 and 5, respectively, as the most urgent patients in ESI level 1 do not abandon, cf. Batt and Terwiesch (2015). Accordingly, the service capacity in our model corresponds to the residual capacity available to patients in ESI levels 2 or higher. We also note that although our primary motivation comes from observable service settings, our models and results are relevant to virtual queues where the information is communicated to customers through delay announcements. In this case, the no information scenario corresponds to only providing (and updating) the total number of customers in the system, and full information corresponds to communicating the exact customer positions in the queue.

To compare these information levels, we study a many-server fluid approximation of the system. The fluid approximation captures the first-order impact of the information and allows for an analytical comparison of system performance with respect to equilibrium average abandonment rates and queue lengths (congestion). In particular, our analysis yields new insights on the interactions between information and key system features, namely priority service and time-varying arrivals. These insights lead to important implications regarding the provision of information in observable priority service systems with time-varying arrivals such as the ED.

1.1. Contributions and Summary of Main Results

Our main contributions and results can be summarized as follows.

1. We study a *novel model of a two-class priority queue with state-dependent abandonment* and time-varying arrivals. The novelty is that the abandonment rates depend on the information level.

2. *Fluid limits and equilibrium analysis.* We derive fluid approximations of the transient dynamics of the system under different information granularity levels and justify them through a Strong Functional Law of Large Number (FSLLN) for the fluid-scaled queue length processes. We further leverage an extended Lyapunov method for time-varying systems to establish the convergence of the fluid limits to asymptotic periodic equilibria under periodic arrival rates. The approach is general and can be used to establish the equilibrium behaviour of other time-varying queueing systems. Further, the results lead to new observations on the impact of information on the equilibrium behavior of the system. In particular, the period of the equilibrium queue length may not be the same as that of the arrival rate, depending on the information level.

3. *Performance comparison under different levels of information.* We leverage our equilibrium results to analytically compare the first-order effects of information under progressively increasing model complexity. To gain insights on the direct and aggregate impact of time-varying arrivals and priority service, we start with a single-class stationary model and then introduce non-stationary arrivals and two priority classes. Our key findings are as follows:

(i) In **single-class stationary** systems, no information minimizes average congestion when customers are sufficiently pessimistic about their queue position (i.e., they assume that they are closer to the end of the queue), by inducing them to abandon sooner, whereas the average abandonment rate is not impacted by the information (Proposition 2). Note that we are thinking of an ED setting here where customers are in a shared waiting room so that it would be difficult for them to infer the service discipline through direct observation of the queue length evolution.

(ii) **Single-class non-stationary** systems that alternate between under- and overloaded regimes have the following information design trade-off between congestion and abandonment. No information minimizes congestion but maximizes abandonment, when customers perceive that they are closer to the end of the queue. Otherwise, providing real-time position information results in lower congestion and higher abandonment (Propositions 3 and 4).

(iii) **Two-priority stationary** systems face the following information design trade-off between the high-priority (HP) and low-priority (LP) classes. Information has the same effect on HP customers as in the single-class case, but may have the *opposite* effect on LP customers (Proposition 5).

(iv) Finally, **two-priority non-stationary** systems are not only subject to the same trade-offs as in simpler systems above, but also face an additional trade-off between the average abandonment of LP and HP classes. This trade-off depends on both, the system load and customers' perceptions of their queue positions. In particular, in a parameter regime relevant to the ED, namely when the HP class alternates between under- and overloaded regimes, if HP customers are sufficiently optimistic (i.e., they perceive a low queue position) and LP customers are sufficiently pessimistic (i.e., they perceive a high queue position), then no information minimizes HP abandonment, whereas full information minimizes LP abandonment (Propositions 7 and 8). Therefore, providing information on the true queue position could increase abandonment for the HP class. Consequently, effective information design for the ED may require providing different information to different priority levels, or adjusting system load through staffing/capacity decisions. We show that, in practically relevant parameter regimes, the trade-off between priority classes can be eliminated by ensuring that the HP class remains underloaded.

4. *Extensions and robustness checks.* Using simulation experiments, we show that our comparative results and insights hold under more general mappings of the system state to customers' perceived queue positions, as well as in stochastic systems with a moderate number of servers.

1.2. Organization of the Paper

The rest of this paper is organized as follows. In §2 we provide a review of the related literature. §3 presents the queueing model. In §4 and §5 we present the fluid limits of the queueing model under the different information levels and establish their convergence to periodic equilibria. In §6, we compare the equilibrium system performance under the three levels of information granularity. In §7 we present the results of our robustness checks. Finally, in §8 we conclude with a discussion of managerial implications and directions for future work.

2. Related Literature

Our work relates to three bodies of literature that we briefly discuss below: Theoretical studies related to delay announcement or state information in service systems; queueing models with customer abandonment; and empirical studies of customer abandonment in service systems.

1. *Theoretical studies related to delay announcements or state information in service systems.* In this body, we group (theoretical) studies that focus on aspects of system information, e.g., its accuracy, equilibrium analysis for a given information scheme, and comparison of performance across different schemes.

(i) There is a large literature on communicating wait-time information in service systems; see Ibrahim (2018) for a recent survey. A primary focus of this body of literature has been to provide a single announcement at the arrival epoch of the customer; see, e.g., Ibrahim et al. (2017), Bassamboo and Ibrahim (2021). A related body of literature focuses on the impact of providing lagged delay information, e.g., Nirenberg et al. (2018), Lakrad et al. (2022) on the performance of the system. Armony et al. (2009) study the impact of fixed and Last-to-Enter-Service (LES) delay announcements on customer behavior in queues with abandonment, by analyzing a fluid approximation and characterizing the resulting system equilibria. In contrast, *we focus on sequentially updated delay information and evaluate the impact of providing queue position information instead.*

(ii) There are also papers that study and compare the effects of different information provision strategies on the *balking* behavior of rational, utility-maximizing customers, and the resulting system performance in terms of throughput, welfare and profit (e.g., Hassin 1986, Chen and Frank 2004, Burnetas and Economou 2007, Guo and Zipkin 2007, 2008, Hassin and Roet-Green 2020). The general takeaway of these studies is that more information may or may not be beneficial. However, *contrary to our model, these papers ignore abandonment and limit attention to stationary single-server FIFO systems.*

(iii) A related stream of work focuses on ticket queues (e.g., Xu et al. 2007, Jennings and Pender 2016, and Kuzu et al. 2019). In ticket queues, customers wait in a virtual queue but are informed of their queue position through a numbered ticket upon arrival. Ticket queues also lead to partial queue information, as some customers may abandon without notifying anyone, and hence render the queue position information on the tickets inaccurate. The literature on ticket queues focuses on understanding the impact of this inaccurate information on system performance. *In contrast, our focus is on comparing performance under different levels of information for priority queues.*

(iv) We model the impact of information on abandonment behaviour through state-dependent (individual) abandonment rates. Earlier studies have used state-dependent abandonment rates to model and compare different delay-announcement strategies. Whitt (1999) and Jouini et al. (2009, 2011) assume that customers react to the delay announcement by balking, but may also subsequently renege if they decide to join. The announcement impacts the queueing performance through the transition rates of the corresponding birth-and-death processes. In our model, the available information impacts the abandonment rates through a general function of the actual state. *In contrast to previous work, we consider a setting where the information (and, hence, the abandonment rates) dynamically changes over time. In addition, we consider time-varying arrivals that cannot be captured through simple birth-and-death processes.*

2. *Queueing models with customer abandonment.* In this body, we group (theoretical) studies that focus on system analysis and optimization, taking the information setting as given. Going back to Barrer (1957), the bulk of these papers take the classical approach of modeling abandonment through an exogenous function of the system state. More recently, there have been several papers that endogenize customers' abandonment behavior, i.e., the abandonment depends not only on the system state, but also on the utility-maximizing decisions of rational, forward-looking customers.

(i) Our work is closer to the large body of literature on queueing models that capture abandonment through exogenous rate functions, see, e.g., Bassamboo and Randhawa (2016), Dong and Ibrahim (2021), and Pender (2017) and the references therein. Closely related to our work are studies that consider state-dependent (individual) abandonment rates. For instance, Whitt (2005a,b) propose and study Markovian queueing models with state-dependent abandonment rates to approximate performance in queueing models with general abandonment distributions. *However, they consider a single-class setting with stationary arrivals and do not examine different levels of information as we do here.*

Our study relies on a fluid approximation of the stochastic queueing system. Fluid models are commonly used to approximate performance in queueing models with abandonment (e.g., Whitt 2006, Liu and Whitt 2011b, Dong et al. 2015, Yu et al. 2021a). We derive fluid approximations for the transient dynamics of the system under time- and state-dependent rates using the strong approximation framework of Mandelbaum et al. (1998). We further establish the existence and study the periodic equilibrium of the fluid models under periodic arrivals. Fluid approximations of the time-dependent equilibrium behaviour of queueing systems are also proposed in Heyman and Whitt (1984) and Liu and Whitt (2011a). Perry and Whitt (2016) and Dong and Perry (2020) also establish the existence of a periodic equilibrium for many-server queues but without abandonment and using ad-hoc approaches. *In contrast, we leverage general methods from the literature on nonlinear dynamical systems (e.g., Khalil 2002) to establish the results.*

(ii) Studies of rational abandonment assume either that customers cannot observe the queue (e.g., Haviv and Ritov 2001, Shimkin and Mandelbaum 2004, Ata et al. 2017, Ata and Peng 2017), or that customers have full information about the queue state (e.g. Hassin 1985, Assaf and Haviv 1990, Afèche and Sarhangian 2015, Cui et al. 2022).

3. *Empirical studies of customer abandonment in service systems.* Several studies have empirically investigated the abandonment behaviour of customers in both unobservable and observable queues.

(i) In the unobservable (virtual) setting, Akşin et al. (2017) and Yu et al. (2017) use structural estimation models to estimate and understand the mechanism through which delay announcements impact callers' abandonment decisions. Yu et al. (2021b) conduct a field experiment in a call center to examine how delay information impact reference-dependent behaviour of customers. Yu et al. (2022) conduct a randomized field experiment in a ride sharing setting to examine the impact of wait time information and its progress on abandonment in virtual queues.

(ii) Closer to our work are empirical studies concerned with observable or "semi-observable" settings. This body of literature has focused on understanding the impact of the visible aspects of the queue, i.e., queue length, position, and service speed on the abandonment behaviour of customers; see Aksin et al. (2022) and the references therein. The majority of the studies are concerned with single class queues, with the exception of Batt and Terwiesch (2015) and Bolandifar et al. (2019) that are concerned with abandonment behaviour of patients in the multiclass setting of EDs. Queue length has been found to be a primary measure with an increasing effect on customer abandonment, even after controlling for wait. This has been observed in retail (deli) queue (Lu et al. 2013) as well as EDs (Batt and Terwiesch 2015, Bolandifar et al. 2019). Buell (2021) finds evidence from grocery queues that queue position - relative to the total length of the queue - also matters. Previous studies, e.g., Janakiraman et al. (2011), have suggested that customer utility is the combination of disutility from remaining wait, and utility from making progress. In addition to queue length, the service speed has also been observed to impact customer behaviour. Aksin et al. (2022) find, using lab experiments, that the sequence of observed service times impact the abandonment behaviour. Batt and Terwiesch (2015) find evidence from the semi-observable setting of an ED that, in addition to queue length and service speed, patients also respond differently to arrivals of patients with higher severity (and hence priority). Bolandifar et al. (2019) find evidence that patients of different severity levels have heterogeneous abandonment responses to observable features of the queue.

In this work, we do not directly model individual customer behaviours. Instead, we model abandonment through a general state-dependent rate function that maps each customer's perception of her position to an abandonment rate. The resulting abandonment behaviour is, however, consistent with the empirical findings on customer abandonment discussed above. In particular, we assume the abandonment rate function is concave and increasing in the queue length. Hence, customers abandon faster from longer queues and making progress at the end of the queue results in a smaller reduction in abandonment probability than for customers who are closer to the head of the queue.

Also, observing faster service times helps customers progress towards lower states faster, and hence reduce their abandonment probabilities. Finally, in the no information setting, we assume that customers are not aware of the priority levels of other customers, but different priority classes may have heterogeneous beliefs about their queue positions. *Our modelling framework allows us to derive new insights on the impact of system dynamics - in particular priority discipline and non-stationary arrivals - on abandonment, queue length and waiting time metrics. These features are prevalent in observable service systems such as EDs. As such, our results are relevant for design of information provision technology (similar to Westphal et al. 2020) and design of future field studies such as Westphal et al. (2022).*

3. Model

We consider a multi-server queueing system with s identical servers and two classes of customers indexed by $k = 1, 2$. Class 1 customers have preemptive priority over class 2 customers; we refer to class 1 as HP (high-priority) and class 2 as LP (low-priority). Customers in the same priority class are served on a First-Come, First-Served (FCFS) basis.

Arrivals to class k follow a Poisson process with rate $\lambda_k(t)$, where $\lambda_k(t)$ is assumed to be bounded and continuous. We assume that the arrival rate is a periodic function, i.e., $\lambda_k(t) = \lambda_k(t + d_k)$, for $t \geq 0$, where $d_k > 0, d_k \in \mathbb{Q}$ is the fundamental (i.e., smallest) period of the arrival rate function for class k. We also examine the special case with stationary arrivals, i.e., with $\lambda_k(t) = \lambda_k$ for all t. Note that, in the case with stationary arrivals, the fundamental period d_k does not exist since $\lambda_k(t) = \lambda_k(t + d_k)$ for arbitrarily small $d_k > 0$. Service times are assumed to be exponentially distributed with class-dependent rates μ_k.

Customers waiting for service abandon the system once their patience expires. A key feature of our model is that a customer's patience varies with the system state and depends on the information design. Under information design I, the patience time (time to abandonment) of a waiting class k customer in position l of her class (i.e., with the l-th earliest arrival time among class k customers) is exponentially distributed with state-dependent rate $\theta(q^I_{kl}(t)/s)$, where $q^I_{kl}(t)$ is the customer's *perceived queue position* at time t, and $\theta(\cdot)$ is a function that maps her perceived position to her abandonment rate. Note that the abandonment rate is determined by applying $\theta(\cdot)$ to the *scaled* perceived queue position, i.e., after dividing it by the number of servers s. Intuitively, scaling the queue position by s captures the effect of system size on customers' abandonment behaviour: as the number of servers (and the system size) increases, we expect the impact of perceiving larger queues to decrease proportionally. We make the following assumptions on $\theta(\cdot)$.

Assumption 1. *The abandonment rate function $\theta(\cdot)$ is continuous, strictly increasing, concave, and bounded, i.e., $\theta(x) \leq M$ for all $x \geq 0$ and some finite constant $M > 0$.*

It is natural to assume that $\theta(\cdot)$ increases in a customer's perceived queue position. The assumption that $\theta(\cdot)$ is concave reflects the notion that the marginal increase in a customer's abandonment rate is smaller the smaller the *relative* increase in her perceived queue position. For example, a customer's abandonment rate increases more if her perceived queue position increases from 10 to 11 (10% relative increase) than from 100 to 101 (1% relative increase).

Customers' perceived queue position depends on the information that they receive. Under full information (F), they are informed about their real-time queue position. Under no information (N), customers are only informed about the real-time total number in the system. Denote by $\{X_k^I(t) : t \geq 0\}$ the process that keeps track of the number of class $k \in \{1, 2\}$ customers in the system under information design $I \in \{F, N\}$.

1. *Full information*: Customers observe the priority classes of all customers as well as their queue positions. Each waiting customer's perceived queue position matches her exact position, i.e., $q_{kl}^F(t) = (\sum_{i=1}^{k-1} X_i^F(t) + l - s)^+$ for $l > (s - \sum_{i=1}^{k-1} X_i^F(t))^+$.

2. *No information*: Customers only observe the total queue length of the system. A class k customer's perceived queue position is determined by the current queue length and a class-dependent relative position fraction $\beta_k \in (0, 1]$, where β_k captures how the customer computes their perceived queue position. Specifically, a class β_k waiting customer's queueing position is given by: $q_{kl}^N(t) = \beta_k (X_1^N(t) + X_2^N(t) - s)^+$ for $l > (s - \sum_{i=1}^{k-1} X_i^N(t))^+$.

Intuitively, β_k reflects a class-level average belief about customers' queue positions during their wait in the absence of queue-position information. Assuming unequal β_k for different priority classes allows us to capture heterogeneity in abandonment responses between priority class. For instance, $\beta_1 < \beta_2$ can be interpreted as high-priority customers having a more optimistic belief about their queue positions. We refer to the no information model with static position fractions as the *β_k model*. In reality, this belief could evolve dynamically as a function of a customer's elapsed waiting time. In Section 7.1 we show that the β_k model serves as an accurate approximation for the model with waiting-time-dependent position fraction.

We also assume the same $\theta(\cdot)$ function for all information levels for analytical tractability and so that we can isolate the interactions between information granularity and system characteristics, e.g., non-stationary arrival rates and priority classes; see also the discussion in Section 8.3.

4. Fluid Approximation

In this section, we obtain a fluid approximation of the queue length process with or without queue position information. To this end, we consider a sequence of queueing systems described in Section 3, indexed by n. The arrival rate and number of servers scale up uniformly in n whereas service rates and the function $\theta(\cdot)$ remain unscaled.

Let s^n and $\lambda_k^n(t)$ denote, respectively, the number of servers and arrival rates in the nth system. Denote by $\{X^{F,n}(t) := (X_1^{F,n}(t), X_2^{F,n}(t)) : t \geq 0\}$ and $\{X^{N,n}(t,\boldsymbol{\beta}) := (X_1^{N,n}(t,\boldsymbol{\beta}), X_2^{N,n}(t,\boldsymbol{\beta})) : t \geq 0\}$ the processes that keep track of the numbers of customers of both classes in the nth system under information design F and N, respectively, where $\boldsymbol{\beta} := (\beta_1, \beta_2)$. Let $A_k \equiv \{A_k(t) : t \geq 0\}$, $S_k \equiv \{S_k(t) : t \geq 0\}$, and $N_k \equiv \{N_k(t) : t \geq 0\}$ be independent unit-rate Poisson processes corresponding to the arrival, service, and abandonment processes, respectively. Then, under full information, the sample path of $X^{F,n}(t)$ is uniquely determined by the initial state $X^{F,n}(0)$ and the following equations:

$$
\begin{aligned}
X_1^{F,n}(t) = X_1^{F,n}(0) &+ A_1\left(\int_0^t \lambda_1^n(u)du\right) - S_1\left(\mu_1 \int_0^t (X_1^{F,n}(u) \wedge s^n)du\right) \\
&- N_1\left(\int_0^t \mathcal{A}_1^{F,n}(X^{F,n}(u))du\right), \qquad (1) \\
X_2^{F,n}(t) = X_2^{F,n}(0) &+ A_2\left(\int_0^t \lambda_2^n(u)du\right) - S_2\left(\mu_2 \int_0^t (X_2^{F,n}(u) \wedge (s^n - X_1^{F,n}(u))^+)du\right) \\
&- N_2\left(\int_0^t \mathcal{A}_2^{F,n}(X^{F,n}(u))du\right), \qquad (2)
\end{aligned}
$$

where $\mathcal{A}_k^{F,n}(X^{F,n}(u))$ denotes the aggregate class k abandonment rate at time u under full information, defined as,

$$
\mathcal{A}_1^{F,n}(X^{F,n}(u)) := \sum_{i=1}^{(X_1^{F,n}(u)-s^n)^+} \theta\left(\frac{i}{s^n}\right), \qquad (3)
$$

$$
\mathcal{A}_2^{F,n}(X^{F,n}(u)) := \sum_{i=(X_1^{F,n}(u)-s^n)^+ + 1}^{(X_1^{F,n}(u)+X_2^{F,n}(u)-s^n)^+} \theta\left(\frac{i}{s^n}\right). \qquad (4)
$$

Similarly, we obtain the sample path of $X^{N,n}(t,\boldsymbol{\beta})$ with initial state $X^{N,n}(0,\boldsymbol{\beta})$ and equations (1)–(2) with $X^{F,n}(t)$ and $\mathcal{A}_k^{F,n}(X^{F,n}(u)$ replaced by $X^{N,n}(t,\boldsymbol{\beta})$ and $\mathcal{A}_k^{N,n}(X^{N,n}(u,\boldsymbol{\beta}))$, where

$$
\mathcal{A}_1^{N,n}(X^{N,n}(u,\boldsymbol{\beta}),\boldsymbol{\beta}) := \theta\left(\frac{\beta_1(X_1^{N,n}(u,\boldsymbol{\beta}) + X_2^{N,n}(u,\boldsymbol{\beta}) - s^n)^+}{s^n}\right)(X_1^{N,n}(u,\boldsymbol{\beta}) - s^n)^+, \qquad (5)
$$

$$
\mathcal{A}_2^{N,n}(X^{N,n}(u,\boldsymbol{\beta}),\boldsymbol{\beta}) := \theta\left(\frac{\beta_2(X_1^{N,n}(u,\boldsymbol{\beta}) + X_2^{N,n}(u,\boldsymbol{\beta}) - s^n)^+}{s^n}\right)\left(X_2^{N,n}(u,\boldsymbol{\beta}) - (s^n - X_1^{N,n}(u,\boldsymbol{\beta}))^+\right)^+. \qquad (6)
$$

For simplicity, we suppress $\boldsymbol{\beta}$ in the expressions of $\mathcal{A}_k^{N,n}(X^{N,n}(t,\boldsymbol{\beta}),\boldsymbol{\beta})$ and $X^{N,n}(t,\boldsymbol{\beta})$ in the remainder of Section 4 and in Section 5 By equations (1) and (2), in the nth system, the number of class k customers at time t, $X_k^{I,n}(t)$, is equal to the initial number-in-system, plus the cumulative number of arrivals, minus the cumulative number of service completions and abandonments until time t. In turn, the aggregate class k service rate at time t equals the individual class k service rate, multiplied by the minimum of the number of available servers for class k customers and the number of class k customers in system.

By equations (3)–(6), the aggregate class k abandonment rate at time t under information level I, $\mathcal{A}_k^{I,n}(X^{I,n}(t))$, equals the sum of the individual abandonment rates of class k customers in queue at time t. That is, $\mathcal{A}_k^{I,n}(X^{I,n}(u)) = \sum_{l=1}^{X_k^{I,n}(t)} \theta(\frac{q_{kl}^I(t)}{s^n})$. Importantly, note that different information designs imply different aggregate abandonment rates for the same system state, $X^{I,n}(u)$.

The following result establishes that the fluid-scaled process $X^{I,n}(t)/n$ (for both information designs) converges to a unique deterministic fluid limit as $n \to \infty$. That is, the fluid limits are good approximations of the corresponding stochastic sample paths when the system is sufficiently large.

THEOREM 1. *Assume that as $n \to \infty$, s^n/n is increasing and $s^n/n \to s$, $\lambda_k^n(t)/n \to \lambda_k(t) < \infty$ uniformly, and $X_k^{I,n}(0)/n \to x_k^I(0)$ almost surely for $k = 1, 2$, $I \in \{F, N\}$. Under information level I, as $n \to \infty$ the scaled process $\{X^{I,n}(t)/n : t \geq 0\}$ converges almost surely to $\{x^I(t) := (x_1^I(t), x_2^I(t)) : t \geq 0\}$ uniformly on compact sets, where $\{x^I(t) : t \geq 0\}$ is the unique solution of the following system of ordinary differential equations starting from initial condition $x^I(0)$:*

$$\dot{x}_1^I(t) = \lambda_1(t) - \mu_1(x_1^I(t) \wedge s) - A_1^I(x^I(t)), \tag{7}$$

$$\dot{x}_2^I(t) = \lambda_2(t) - \mu_2\left((s - x_1^I(t))^+ \wedge x_2^I(t)\right) - A_2^I(x^I(t)), \tag{8}$$

where,

$$A_1^I(x(t)) = \begin{cases} \int_0^{(x_1(t)-s)^+} \theta(u/s)\, du, & \text{if } I = F, \\ \theta\left(\frac{\beta_1(x_1(t)+x_2(t)-s)^+}{s}\right)(x_1(t) - s)^+, & \text{if } I = N; \end{cases} \tag{9}$$

$$A_2^I(x(t)) = \begin{cases} \int_{(x_1(t)-s)^+}^{(x_1(t)+x_2(t)-s)^+} \theta(u/s)\, du, & \text{if } I = F, \\ \theta\left(\frac{\beta_2(x_1(t)+x_2(t)-s)^+}{s}\right)(x_2(t) - (s - x_1(t))^+)^+, & \text{if } I = N. \end{cases} \tag{10}$$

Note that for simplicity we suppress the argument $\boldsymbol{\beta}$ in $x_1^N(t, \boldsymbol{\beta})$ and $A_k^N(x(t), \boldsymbol{\beta})$ in Sections 4 and 5. The proof of Theorem 1 is provided in Appendix A. The proof is based on verifying the conditions of Theorem 2.2 of Mandelbaum et al. (1998) which establishes a Functional Strong Law of Large Numbers (FSLLN) for a general family of queueing processes. In our model, the main difficulty is to establish the Lipschitz continuity of the abandonment rate functions. Furthermore, in the case of full information, the abandonment rates are different for individual customers in the same priority class. This introduces additional technical difficulties when showing the convergence of the summations for the system abandonment rate in (3)–(6) to the integrals in (9) and (10). The assumption that s^n/n is increasing is made to facilitate this step of the proof.

5. Equilibrium Analysis

In this section, we study the long-run behavior of the fluid models developed in Section 4.

Definition 1. A solution $\tilde{x}^I := \{\tilde{x}^I(t) : t \geq 0\}$ to the system of ODEs (7)–(8) under a fixed information level I is a **periodic equilibrium** if there exists a vector $(p_1, p_2) \in \mathbb{R}_+^2$ such that $\tilde{x}_k^I(t + p_k) = \tilde{x}_k^I(t)$ for all $t \geq 0$ and $k = 1, 2$. The smallest (p_1, p_2) pair (if it exists) is referred to as the **fundamental period** of the equilibrium.

Remark 1. Note that, when $\tilde{x}_k^I(t + p_k) = \tilde{x}_k^I(t)$ holds for arbitrarily small p_k, i.e., $\tilde{x}_k^I(t) = \tilde{x}_k^I$ is a constant for some $k \in \{1, 2\}$, the fundamental period of the equilibrium does not exist. When $\tilde{x}_k^I(t)$ is a constant for both $k = 1, 2$, then the periodic equilibrium reduces to an equilibrium point.

Our first result establishes the existence of a unique periodic equilibrium under the assumption that the service rates for the two classes are the same. Recall that $\lambda_k(t)$ has fundamental period d_k and denote by $d := lcm(d_1, d_2)$ the least common multiple of d_1, d_2 (such d must exist since $d_k \in \mathbb{Q}$). Then, d is a period of the total arrival rate of the system, i.e., $\Lambda(t) := \lambda_1(t) + \lambda_2(t)$, although it may not be its fundamental period. In particular, the fundamental period of $\Lambda(t)$ does not exist when $\Lambda(t)$ is static (see case (ii) of Example 1), and can be either equal to or smaller than d.

Proposition 1. *Assume that $\mu_1 = \mu_2 = \mu$ and $\beta_1 = \beta_2 = \beta$. Under any information level $I \in \{F, N\}$, (7) and (8) have a unique periodic equilibrium $\tilde{x}^I$ with period (p_1^I, p_2^I), where $(p_1^F, p_2^F) = (d_1, d)$ and $(p_1^N, p_2^N) = (d, d)$.*

We prove Proposition 1 by establishing the existence of a fixed point of the Poincaré map (see Definition 4) with respect to the system of two-dimensional ODEs (7)–(8). The approach is general and can be adapted to establish the existence and uniqueness of a periodic equilibrium for other time-varying queueing models. Since the monotonicity of Poincaré map (see Proposition 9) only applies to one-dimensional ODEs, we assume equal service rates to convert our two-dimensional ODE to a one-dimensional one in terms of the total queue length $x_1 + x_2$ for $x = (x_1, x_2) \in \mathbb{R}^2$. The assumption of equal service rates and class-independent β is however not necessary for the existence of the equilibrium. Numerical experiments suggest that Proposition 1 continues to hold if $\mu_1 \neq \mu_2$ and $\beta_1 \neq \beta_2$ (see Example 1). Further, we note that under stationary arrivals, the periodic equilibrium reduces to a single equilibrium point $\tilde{x}^I \in \mathbb{R}_+^2$. In this case, under each information level I, Proposition 1 implies that there exists a unique equilibrium point $\tilde{x}^I$.

By (7)–(10), the HP number-in-system process under full information solely depends on the HP arrival rate, $\lambda_1(t)$, whereas the HP and LP number-in-system processes under no information and the LP number-in-system process under full information depend on the arrival rates of both classes. As a result, p_1^F is determined by the period of the HP arrival rate, d_1, while p_1^N, p_2^I, for $I \in \{F, N\}$, are determined by the period of the total arrival rate, d.

Note that (p_1^I, p_2^I) is a period, but not necessarily the fundamental period of the periodic equilibrium $\tilde{x}^I$. However, numerically we observe instances where (p_1^I, p_2^I) is the fundamental period,

(a) HP number-in-system, $p_1^F = 10, p_1^N = 30$. (b) LP number-in-system, $p_2^F = p_2^N = 30$.

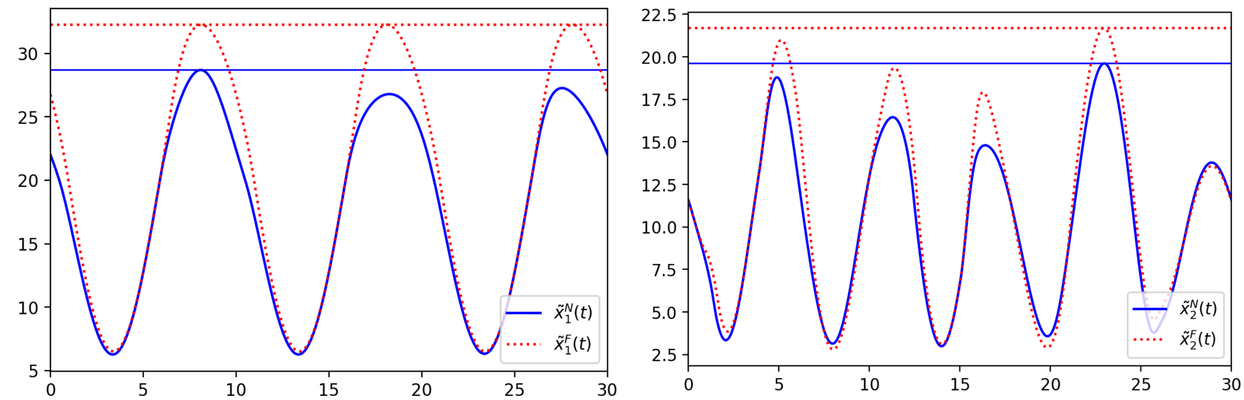

Figure 1 **Trajectories of number-in-system processes in periodic equilibrium (**$\mu_1 = 1, \mu_2 = 2,\ \beta_1 = 0.8, \beta_2 = 0.9$, $s = 20, \theta(x) = 4.2 - 4e^{-x}, \lambda_1(t) = 20(1 - 0.8\sin(\pi t/5)), \lambda_2(t) = 20(1 - 0.8\sin(\pi t/3))$).

which implies that the period of the equilibrium could indeed depend on the information level. That is, information could have a first-order impact on the dynamics of the system by changing the period of its equilibrium. This is in contrast with other examples of fluid models in the literature, where the equilibria of the trajectories coincide with that of the arrival rate (e.g., Heyman and Whitt 1984, Dong and Perry 2020). We illustrate this numerically in the following example.

Example 1. Consider the system with $s = 20$, $\mu_1 = 1, \mu_2 = 2$, $\beta_1 = 0.8, \beta_2 = 0.9$, $\theta(x) = 4.2 - 4e^{-x}$. In this case, $\mu_1 < \mu_2$, $\beta_1 < \beta_2$, and $\theta(0) = 0.2 < \mu_2$. Consider the following two sets of sinusoidal arrival rates: (i) $\lambda_1(t) = 20(1 - 0.8\sin(\pi t/5))$, $\lambda_2(t) = 20(1 - 0.8\sin(\pi t/3))$; and (ii) $\lambda_1(t) = 20(1 - 0.8\sin(\pi t/12))$, $\lambda_2(t) = 20(1 + 0.8\sin(\pi t/12))$. For each of cases (i) and (ii), the trajectories of number-in-system converges to a periodic equilibrium $(\tilde{x}_1^I, \tilde{x}_2^I)$ for each information level I. In particular,

(i) If $\lambda_1(t) = 20(1 - 0.8\sin(\pi t/5))$ and $\lambda_2(t) = 20(1 - 0.8\sin(\pi t/3))$. Then $(d_1, d_2) = (10, 6)$ and $d = 30$, and the periodic equilibrium $(\tilde{x}_1^I, \tilde{x}_2^I)$ has a fundamental period $(p_1^I, p_2^I) = (30, 30)$, for $I = N$, and $(p_1^I, p_2^I) = (10, 30)$, for $I = F$, as shown in Figure 1. This illustrates that the fundamental period of the equilibrium can depend on the information design.

(ii) If $\lambda_1(t) = 20(1 - 0.8\sin(\pi t/12))$ and $\lambda_2(t) = 20(1 + 0.8\sin(\pi t/12))$. Then $d_1 = d_2 = d = 24$, and the periodic equilibrium $(\tilde{x}_1^I, \tilde{x}_2^I)$ has a fundamental period $(p_1^I, p_2^I) = (24, 24)$ for $I \in \{F, N\}$, as shown in Figure 2. Note that, in this case, the total arrival rate $\Lambda(t) = 40$ is a constant. This illustrates that the fundamental period of the equilibrium may not coincide with the period of the total arrival process.

Next, we investigate whether the fluid model is *asymptotically periodic*, i.e., whether starting with any initial condition the trajectories converge to the periodic equilibrium $\tilde{x}^I$ as $t \to \infty$. To this end, we examine the stability of the periodic equilibrium. Let $x^I(t)$ be the unique solution to the system

(a) HP number-in-system, $p_1^I = 24$. (b) LP number-in-system, $p_2^I = 24$.

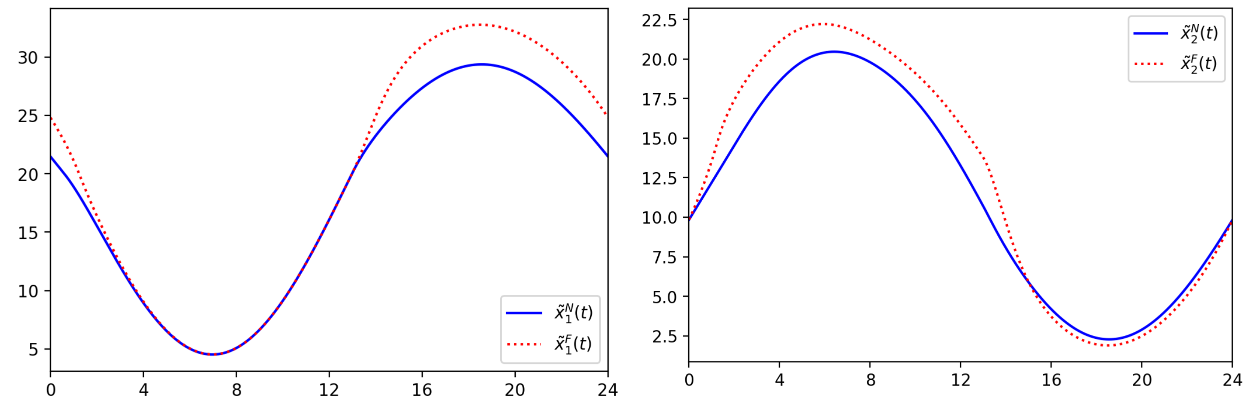

Figure 2 **Trajectories of number-in-system processes in periodic equilibrium** ($\mu_1 = 1, \mu_2 = 2,\ s = 20,$ $\beta_1 = 0.8, \beta_2 = 0.9$

, $\theta(x) = 4.2 - 4e^{-x}, \lambda_1(t) = 20(1 + 0.8\sin(\pi t/12)), \lambda_2(t) = 20(1 - 0.8\sin(\pi t/12))$).

of ODEs (7)–(8) under information level I with initial condition $x^I(0)$. Denote by $f_1^I(t, x^I(t))$ and $f_2^I(t, x^I(t))$ the RHS of equations (7) and (8). That is, $f_k^I(t, x^I(t))$ denotes the net flow rate of class k customers under information level I, at time t and state $x^I(t)$. Let $y_k^I(t) := x_k^I(t) - \tilde{x}_k^I(t)$ denotes the deviation of the trajectory of the number-in-system process $x_k^I(t)$ from the periodic equilibrium $\tilde{x}_k^I(t)$, for $k = 1, 2$, and $I \in \{F, N\}$. Consider the following system:

$$\dot{y}_1^I(t) = f_1^I(t, y + \tilde{x}^I) - f_1^I(t, \tilde{x}^I) =: \tilde{g}_1^I(t, y), \tag{11}$$

$$\dot{y}_2^I(t) = f_2^I(t, y + \tilde{x}^I) - f_2^I(t, \tilde{x}^I) =: \tilde{g}_2^I(t, y). \tag{12}$$

Note that although $\lambda_k(t)$ in $f_k^I(t, y + \tilde{x}^I)$ and $f_k^I(t, \tilde{x}^I)$ cancel out, the system (11)–(12) is not time-invariant since g_k^I depends on the the time-varying trajectory $\tilde{x}^I(t)$. Moreover, the solution of (7)–(8) depends on the initial state via $\lambda(0)$.

Definition 2. $(0, 0)$ is an **equilibrium point** of $\dot{y} = g(t, y)$ if $g(t, 0) = 0$, for $t \geq 0$.

Observe that $y = (0, 0)$ is an equilibrium point for system (11)–(12) since $\tilde{g}_1^I(t, 0) = 0$ and $\tilde{g}_2^I(t, 0) = 0$. The following definition formalizes the notion of stability for time-varying trajectories.

Definition 3. Let $g(t, y)$ be a Lipschitz function defined on $\mathbb{R}_+ \times \mathbb{R}^2$, and $g(t, 0) = 0$. The equilibrium point $y = (0, 0)$ of $\dot{y} = g(t, y)$ is **globally uniformly asymptotically stable** if for any initial condition $y(0)$, $\lim_{t\to\infty} |y(t)| = 0$, where $|\cdot|$ is the standard Euclidean norm.
That is, if the origin is a globally asymptotically stable equilibrium point of a system, then a trajectory starting from an arbitrary point converges to the origin as t tends to infinity.

Theorem 2. *Assume that $\theta(0) > 0$, $\mu_1 = \mu_2 = \mu$, and $\beta_1 = \beta_2 = \beta$. Then $(y_1, y_2) = (0, 0)$ is a globally uniformly asymptotically stable equilibrium for (11)–(12) under each information level $I \in \{F, N\}$.*

Theorem 2 implies that, under any of the information levels, the corresponding fluid model converges to its periodic equilibrium as $t \to \infty$. In the case of stationary arrivals, since the periodic equilibrium reduces to a single equilibrium point $\tilde{x}^I \in \mathbb{R}^2_+$, the theorem implies that starting from any initial condition, the trajectories converge to $\tilde{x}^I$ as $t \to \infty$.

The proof of Theorem 2 relies on an extended Lyapunov method (see Theorem 6 in Appendix B.3). Lyapunov methods are commonly used to prove the asymptotic stability of stationary systems, where one needs to find a positive definite Lyapunov function $V(y)$ for trajectory $y \in \mathbb{R}^n$ with a negative definite derivative $\dot{V}(y)$. For time-varying systems, the stability of the equilibrium point, in general, depends on (the initial) time. Therefore, one needs to find Lyapunov function candidates $V(t, y)$ on $\mathbb{R}^+ \times \mathbb{R}^n$. This requires satisfying more strict conditions for the positive definiteness of $V(t, y)$ to hold and the natural candidates typically used for stationary systems in the literature fail to satisfy the conditions. Therefore, to facilitate the proof, we assume positive abandonment rates (to bound $\dot{V}(t, y)$), equal service rates and class-independent β (to bound or cancel out the cross-product terms in $\dot{V}(t, y)$). Nevertheless, we find numerically that these assumptions are not necessary for the stability of the equilibrium (see Example 1).

Remark 2. To prove Proposition 1 and Theorem 2 we introduce and apply general results from the literature on nonlinear dynamical systems. As such, the methods we use here can be adapted to examine the long-run behaviour of other time-varying queueing models as well.

6. Performance Comparisons

In this section, we study how information impacts the system's equilibrium performance measures, namely, the average number of customers in the system and the average abandonment rate.

6.1. Overview of Main Results and Analysis Roadmap

Our key findings focus on non-stationary two-priority systems with alternating under-/overload. Figure 3 provides a visual summary of these findings:

1. *Trade-off between HP number-in-system and abandonment rate:* Compared to full information (F), no information (N) yields a larger number-in-system but a smaller abandonment rate[1], if β_1 is below some threshold, and vice versa for larger β_1; see Figures 3(a) and 3(d). These results also hold for customers in single-class systems.

2. *Trade-off or alignment between LP number-in-system and abandonment rate:* For LP customers, information design involves a similar trade-off between queueing and abandonment for some $\boldsymbol{\beta}$ ranges, whereas no information minimizes *both*, number-in-system and abandonment, if HP and LP customers are sufficiently pessimistic, e.g., $\beta_1 \gtrapprox 0.4$, $\beta_2 \gtrapprox 0.6$; see Figures 3(b) and 3(e).

[1] Under stationary demand, the abandonment rate is invariant to the information design.

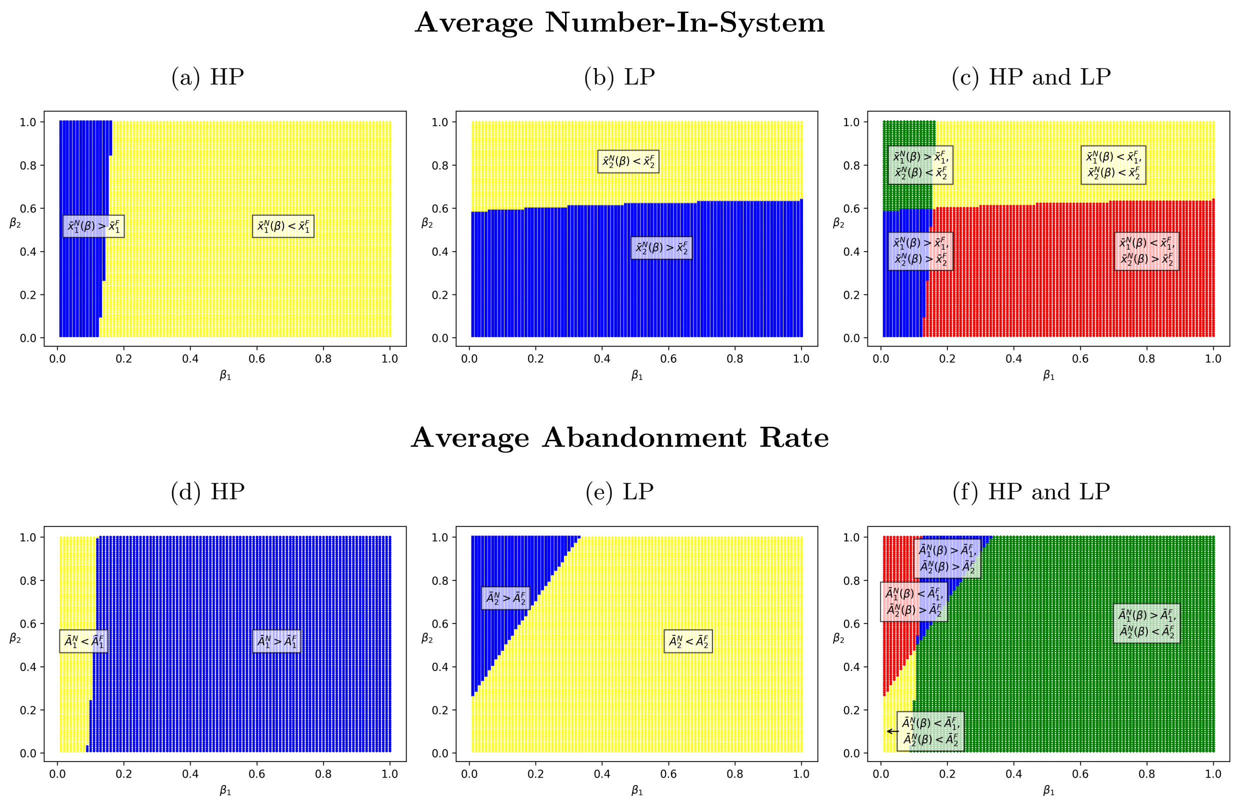

Figure 3 **Non-stationary two-priority system: Ranking of equilibrium performance metrics under information regimes N and F, as function of $\boldsymbol{\beta}$**

$\mu_1 = \mu_2 = 1,\ s = 100,\ \rho_1 = 0.9,\ \rho_2 = 0.5,\ \ \lambda_k(t) = s\rho_k(1 - 0.5\sin(\pi t/12)),\ \theta(x) = 2 - e^{-x}.$

3. *Trade-offs between HP and LP performance.* Information may have consistent or *opposite* effects on the number-in-system (Figure 3(d)) and the abandonment rate (Figure 3(f)) of the HP and LP classes. Information has consistent performance effects for $\boldsymbol{\beta}$ in the blue and yellow areas, but is subject to trade-offs for $\boldsymbol{\beta}$ in the red and green areas.

Deriving and explaining these comparison results for a two-class system with non-stationary arrival rates poses significant challenges, particularly in the most practically relevant case when the HP load alternates between over- and under-loaded. Therefore, to highlight the individual and collective impact of prioritization and time-varying arrivals on the effects of information provision, we develop and present these results in the following sequence: In Section 6.2, we summarize preliminaries. In Section 6.3, we start with the single-class model with stationary arrivals. In Section 6.4, we study the impact of non-stationary arrivals for a single customer class. In Section 6.5, we study the impact of priority service by considering two priority classes with stationary arrival rates. In Section 6.6, we consider both time-varying arrivals and two priority classes. Figure 4 shows this analytical roadmap for Sections 6.3-6.6 and the results and trade-offs we identify along the way.

In Section 8.1 we recap these results; in Section 8.2 we highlight some managerial implications.

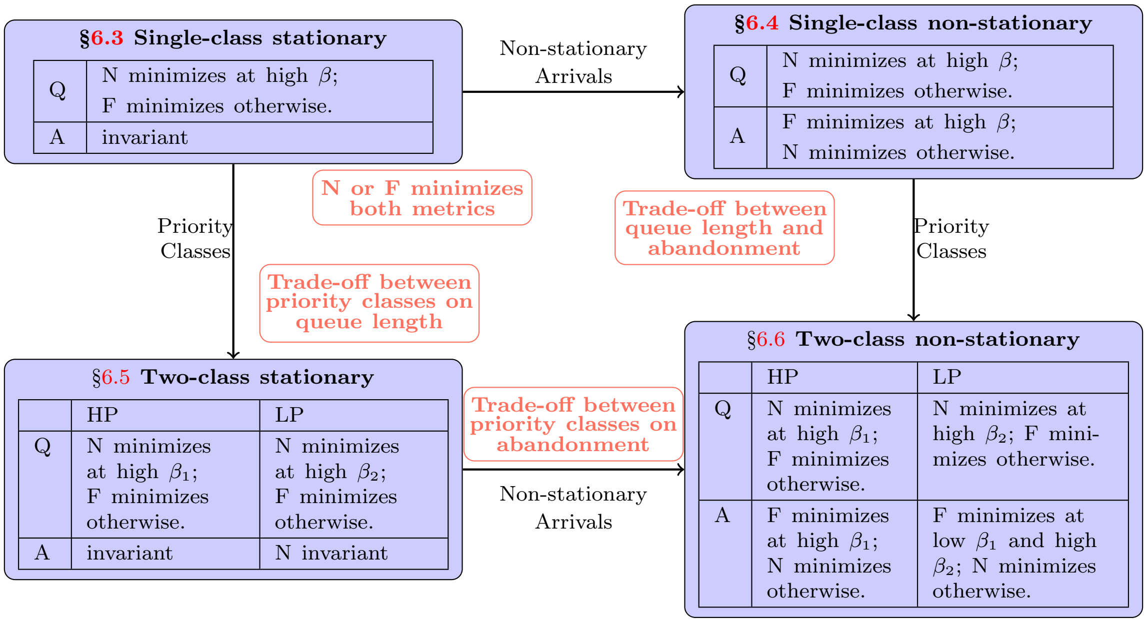

Figure 4 **Analytical roadmap for Sections 6.3-6.6 and summary of performance comparisons under no (N) versus full (F) information (Q = average number-in-system, A = average abandonment rate).**

6.2. Preliminaries

Recall that $\tilde{x}^I$ denotes the periodic equilibrium number-in-system process under information level I. Let $\bar{x}_k^I$ denote the time-average number-in-system and $\bar{A}_k^I$ the time-average system abandonment rate. Under stationary arrivals, the process $\tilde{x}^I$ is constant over time so that $\bar{x}_k^I = \tilde{x}_k^I$ and $\bar{A}_k^I = A_k^I(\bar{x}_k^I)$. Under non-stationary arrivals, Proposition 1 and the definition of d imply that d is a period of $\tilde{x}_k^I$, for $k = 1,2$, so that $\bar{x}_k^I = \frac{1}{d}\int_0^d \tilde{x}_k^I(t)dt$ and $\bar{A}_k^I = \frac{1}{d}\int_0^d A_k^I(\tilde{x}_k^I(t))dt$. Let $\rho_k(t) := \lambda_k(t)/s\mu_k$ be the traffic intensity (load) of class k customers at time t, and let $\rho_k := \frac{1}{d}\int_0^d \rho_k(t)dt$, $\underline{\rho}_k := \min_{t\geq 0}\rho_k(t)$, and $\bar{\rho}_k := \max_{t\geq 0}\rho_k(t)$ be the average, minimum, and maximum class k load, respectively.

To develop the results, we express the time-average system abandonment rate, $\bar{A}_k^I$, in two ways. First, we write $\bar{A}_k^I$ as the difference between the time averages of the system arrival rate and the system service rate during an interval of length d. This holds because by Proposition 1, the in- and outflows are balanced during such an interval, so (7) and (8) yield the following equations:

$$\bar{A}_1^I = \frac{1}{d}\int_0^d \lambda_1(t)dt - \frac{\mu_1}{d}\int_0^d (\tilde{x}_1^I(t)\wedge s)dt \text{ for } I\in\{F,N\}, \tag{13}$$

$$\bar{A}_2^I = \frac{1}{d}\int_0^d \lambda_2(t)dt - \frac{\mu_2}{d}\int_0^d \left((s-\tilde{x}_1^I(t))^+ \wedge \tilde{x}_2^I(t)\right)dt \text{ for } I\in\{F,N\}. \tag{14}$$

Second, we write $\bar{A}_k^I$ by averaging the sum of individual customers' abandonment rates, given by (9) and (10), over a time interval of length d. These individual customer abandonment rates depend on the system's information level. This approach yields the following equations:

$$\bar{A}_1^N := \frac{1}{d}\int_0^d \theta\left(\beta_1\frac{(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s)^+}{s}\right)(\tilde{x}_1^N(t)-s)^+dt, \tag{15}$$

$$\bar{A}_1^F := \frac{1}{d}\int_0^d \int_0^{(\tilde{x}_1^F(t)-s)^+} \theta\left(u/s\right) du dt, \tag{16}$$

$$\bar{A}_2^N := \frac{1}{d}\int_0^d \theta\left(\beta_2 \frac{(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s)^+}{s}\right)(\tilde{x}_2^N(t)-(s-\tilde{x}_1^N(t))^+)^+ dt, \tag{17}$$

$$\bar{A}_2^F := \frac{1}{d}\int_0^d \int_{(\tilde{x}_1^F(t)-s)^+}^{(\tilde{x}_1^F(t)+\tilde{x}_2^F(t)-s)^+} \theta\left(u/s\right) du dt. \tag{18}$$

In what follows, we refer to equations (13)-(18) to help discuss the results and the underlying intuition. Further, recall that we omitted $\boldsymbol{\beta}$ from the notations for systems with no information in the preceding sections. Since the comparison results between the no and full information models depend on $\boldsymbol{\beta}$, we will use complete notations with $\boldsymbol{\beta}$ as a dependent variable for results in the remaining section. That is, we replace $\tilde{x}_k^N(t)$, $\bar{x}_k^N$, and $\bar{A}_k^N$ with $\tilde{x}_k^N(t,\boldsymbol{\beta})$, $\bar{x}_k^N(\boldsymbol{\beta})$, and $\bar{A}_k^N(\boldsymbol{\beta})$, respectively, when presenting the comparison results in Sections 6.3–6.6.

6.3. Single Class with Stationary Arrivals

In this section, we consider a single class of customers with stationary arrivals. We omit the customer class subscript for simplicity. Under stationary arrivals, the equilibrium is constant over time, so that (13), (15) and (16) reduce, respectively, to:

$$\bar{A}^I = \lambda - \mu(\bar{x}^I \wedge s), \text{ for } I \in \{F, N\}, \tag{19}$$

$$\bar{A}^N := \theta\left(\beta \frac{(\bar{x}^N - s)^+}{s}\right)(\bar{x}^N - s)^+, \tag{20}$$

$$\bar{A}^F := \int_0^{(\bar{x}^F - s)^+} \theta\left(u/s\right) du. \tag{21}$$

These equations imply the following rankings of performance measures (we omit a formal proof).

Proposition 2. *For single-class systems with stationary arrivals, the equilibrium average number-in-system and abandonment rate under no (N) and full (F) information compare as follows:*

1. *If $\rho \le 1$, then $\bar{x}^N(\beta) = \bar{x}^F = s\rho$ and $\bar{A}^N(\beta) = \bar{A}^F = 0$.*
2. *If $\rho > 1$, then $s < \bar{x}^N(\beta), s < \bar{x}^F$ and $\bar{A}^N(\beta) = \bar{A}^F = \lambda - s\mu$. There is a threshold $\beta^* \in (0,1)$ such that $\bar{x}^N(\beta^*) = \bar{x}^F$ and:*
   (a) *If $\beta \in [0, \beta^*)$, then $\bar{x}^F < \bar{x}^N(\beta)$.*
   (b) *If $\beta \in (\beta^*, 1]$, then $\bar{x}^N(\beta) < \bar{x}^F$.*

Proposition 2.2 highlights how the effect of information in overloaded systems ($\rho > 1$) depends on the position factor β. (Information has no effect in underloaded systems, as there is no queue.) There exists a (load-dependent) threshold β^* such that no information (N) yields a longer queue than full information (F) if $\beta < \beta^*$, and vice versa if $\beta > \beta^*$. Intuitively, this holds as a result of the following three factors: (i) The equilibrium abandonment rate is invariant to the information under stationary arrivals, see (19); (ii) the system abandonment rate under both information regimes increases in the queue length; and (iii) for any *fixed* queue length, the parameter β in the no-information regime has the following effects on abandonment: the individual and system abandonment rates are continuously increasing in β, lower compared to full information (F) for $\beta = 0$, and higher compared to F for $\beta = 1$; see (20) and (21). Therefore, for β below (above) the threshold β^*, no information yields a longer (shorter) queue of customers who abandon at a slower (faster) average rate, compared to customers under full information.

In sum, in a stationary overloaded single-class system, the goal of effective information design is to shorten the queue by inducing customers to abandon sooner. In this sense, by Proposition 2.2 no information is less effective than full information for small β, and more effective for large β.

6.4. Single Class with Non-Stationary Periodic Arrivals

We turn to the effects of non-stationary arrivals in single-class systems. Compared to the stationary case, time-varying arrivals give rise to one more load regime, whereby the system alternates between over- and underloaded, that is, $\underline{\rho} < 1 < \bar{\rho}$. This load regime is prevalent in practice and yields a key difference, compared to the stationary setting: a trade-off between queueing and abandonment, as shown below in Propositions 3 and 4.

Proposition 3 establishes the following ranking of the *average* equilibrium number-in-system.

Proposition 3. *For single-class systems with non-stationary periodic arrivals, the equilibrium average number-in-system under no (N) and full (F) information compare as follows:*

1. *If* $\bar{\rho} \leq 1$, *then* $\bar{x}^N = \bar{x}^F$.
2. *If* $\bar{\rho} > 1$ *and* $\max_{t \geq 0} \tilde{x}^N(t, 0) > s$, *there is a threshold beta*$_q^* \in (0, 0.5)$ *such that* $\bar{x}^N(\beta_q^*) = \bar{x}^F$ *and:*
   - (a) *If* $\beta \in [0, \beta_q^*)$, *then* $\bar{x}^F < \bar{x}^N(\beta)$.
   - (b) *If* $\beta \in (\beta_q^*, 1]$, *then* $\bar{x}^N(\beta) < \bar{x}^F$.

Proposition 3 builds on stronger results that rank the equilibrium number-in-system processes $\tilde{x}^N(t, \beta)$ and $\tilde{x}^F(t)$ for *all* t (see Lemma 4 in Appendix C.3). Proposition 3 shows that the corresponding ranking results for stationary systems (Proposition 2) are robust and naturally generalize to the non-stationary case, specifically in the important case where the system is overloaded at least some of the time ($\bar{\rho} > 1$, Part 2): Compared to full information, no information increases the *average* equilibrium number-in-system for small β, and weakly reduces this metric for large β.

Proposition 3.2 also adds an insight that is specific to non-stationary systems: The information design impacts the average equilibrium number-in-system even if the system is *underloaded on average* (i.e., $\rho < 1$), so long as a queue forms some of the time under no information with $\beta = 0$ (i.e., if $\max_{t \geq 0} \tilde{x}^N(t, 0) > s$).

Proposition 4 establishes the following ranking of the *average* system abandonment rates.

Proposition 4. *For single-class systems with non-stationary periodic arrivals, the equilibrium average abandonment rates under no (N) and full (F) information compare as follows:*

1. *If* $\bar{\rho} \le 1$*, or* $\underline{\rho} \ge 1$*, or more generally* $\min_{t\ge 0} \tilde{x}^N(t,1) \ge s$*, then* $\bar{A}^N(\beta) = \bar{A}^F$*.*
2. *If* $\underline{\rho} < 1 < \bar{\rho}$ *and* $\max_{t\ge 0} \tilde{x}^N(t,0) > s > \min_{t\ge 0} \tilde{x}^N(t,1)$*, there is a threshold* $\beta_a^* \in (0, 0.5)$ *such that* $\bar{A}^N(\beta_a^*) = \bar{A}^F$ *and:*
   (a) *If* $\beta \in [0, \beta_a^*)$*, then* $\bar{A}^N(\beta) < \bar{A}^F$*.*
   (b) *If* $\beta \in (\beta_a^*, 1]$*, then* $\bar{A}^N(\beta) > \bar{A}^F$*.*

Comparing Propositions 3.2 and 4.2 with Proposition 2 shows that non-stationary arrivals yield one key difference compared to stationary settings: A trade-off between queueing and abandonment in the practically most relevant regime where the system alternates between under- and overloaded ($\underline{\rho} < 1 < \bar{\rho}$):[2] For $\beta < \min(\beta_a^*, \beta_q^*)$ the no-information downside of higher congestion (a longer queue length) is offset by throughput gain, i.e., a lower abandonment rate; for $\beta > \max(\beta_a^*, \beta_q^*)$ this trade-off is reversed, as full information yields more congestion and higher throughput.

Intuitively, this trade-off follows because the information regime with the larger number-in-system also experiences a higher average server utilization, implying that more customers are served and fewer abandon, compared to the information regime with the smaller number-in-system.

In sum, information design in non-stationary systems that alternate between under- and overloaded is subject to a trade-off between two conflicting key objectives, minimizing congestion (queueing) and maximizing throughput (minimizing abandonment). The key implication is that the information design must carefully consider and balance the impact on *both* performance measures, queue length and abandonment, as well as the resulting costs and benefits. For example, with relatively more pessimistic customers (i.e., larger β), no information is preferable to full information if the resulting throughput loss and/or the cost of abandonment per customer are relatively small.

6.5. Two-Class Priority System with Stationary Arrivals

We now consider systems with two priority classes, starting with the case of stationary arrivals. The interplay between two priority classes yields a key difference, compared to the single-class setting: The information design may have *opposite* effects on the queue lengths of the two classes. Building on the balance equations (13)-(18), Proposition 5 makes these effects precise by ranking for each class the equilibrium number-in-system under no (N) vs. full (F) information, as function of the system loads. Information does not affect the abandonment rates in stationary systems.

Proposition 5. *For two-priority systems with stationary arrivals, the equilibrium average numbers-in-system and abandonment rates under no (N) and full (F) information compare as follows:*

[2] The thresholds β_q^* and β_a^* are close but do not necessarily coincide; either threshold may be larger, and their ranking depends on the abandonment rate function and system loads.

1. *If* $\rho_1 \leq 1$*, then* $\bar{x}_1^N(\boldsymbol{\beta}) = \bar{x}_1^F = s\rho_1$ *and* $\bar{A}_1^N(\boldsymbol{\beta}) = \bar{A}_1^F = 0$*.*

For LP customers:

   (a) *If* $\rho_1 + \rho_2 \leq 1$*, then* $\bar{x}_2^N(\boldsymbol{\beta}) = \bar{x}_2^F = s\rho_2$ *and* $\bar{A}_2^N(\boldsymbol{\beta}) = \bar{A}_2^F = 0$*.*

   (b) *If* $\rho_1 + \rho_2 > 1$*, then* $\bar{A}_2^N(\boldsymbol{\beta}) = \bar{A}_2^F = \lambda_2 - s(1-\rho_1)\mu_2$*, and there exists a threshold* $\beta_2^* \in (0,1)$ *such that* $\bar{x}_2^N(\boldsymbol{\beta}) = \bar{x}_2^F$ *and:*

      (i) If $\beta_2 \in [0, \beta_2^*)$ *then* $\bar{x}_2^F < \bar{x}_2^N(\boldsymbol{\beta})$*.*

      (ii) If $\beta_2 \in (\beta_2^*, 1]$*, then* $\bar{x}_2^N(\boldsymbol{\beta}) < \bar{x}_2^F$*.*

2. *If* $\rho_1 > 1$*, then* $\bar{x}_1^N(\boldsymbol{\beta}) > s$*,* $\bar{x}_1^F > s$*,* $\bar{A}_1^N(\boldsymbol{\beta}) = \bar{A}_1^F = \lambda_1 - s\mu_1$*, and* $\bar{A}_2^N(\boldsymbol{\beta}) = \bar{A}_2^F = \lambda_2$*.*

There exist two increasing threshold functions $\beta_1^*(\beta_2) \in (0, 0.5)$ *and* $\beta_2^*(\beta_1) \in (0,1]$*, which are decreasing in the LP load* ρ_2*, and partition the parameter space of* $\boldsymbol{\beta}$ *into four regions:*

   (a) *If* $\beta_1 \in [0, \beta_1^*(\beta_2))$ *and* $\beta_2 \in [0, \beta_2^*(\beta_1))$*, then* $\bar{x}_1^N(\boldsymbol{\beta}) > \bar{x}_1^F$ *and* $\bar{x}_2^N(\boldsymbol{\beta}) > \bar{x}_2^F$*.*

   (b) *If* $\beta_1 \in [0, \beta_1^*(\beta_2))$ *and* $\beta_2 \in (\beta_2^*(\beta_1), 1]$*, then* $\bar{x}_1^N(\boldsymbol{\beta}) > \bar{x}_1^F$ *and* $\bar{x}_2^N(\boldsymbol{\beta}) < \bar{x}_2^F$*.*

   (c) *If* $\beta_1 \in (\beta_1^*(\beta_2), 1]$ *and* $\beta_2 \in [0, \beta_2^*(\beta_1))$*, then* $\bar{x}_1^N(\boldsymbol{\beta}) < \bar{x}_1^F$ *and* $\bar{x}_2^N(\boldsymbol{\beta}) > \bar{x}_2^F$*.*

   (d) *If* $\beta_1 \in (\beta_1^*(\beta_2), 1]$ *and* $\beta_2 \in (\beta_2^*(\beta_1), 1]$*, then* $\bar{x}_1^N(\boldsymbol{\beta}) < \bar{x}_1^F$ *and* $\bar{x}_2^N(\boldsymbol{\beta}) < \bar{x}_2^F$*.*

Underloaded HP class (Proposition 5.1): In this case HP customers do not queue and utilize $s\rho_1$ servers. As a result, for LP customers the system is equivalent to a single-class stationary system with $s(1-\rho_1)$ servers. Therefore, the results of Proposition 5.1 for LP customers are consistent with Proposition 2 for the single-class case: Compared to full information (F), no information (N) yields a larger (smaller) number of LP customers for β_2 below (above) some threshold.

Overloaded HP class (Proposition 5.2): If the system is overloaded with HP customers, more information may have the *same or opposite* effect on the HP and LP queue lengths. Before elaborating on these results, we note that, whereas no LP customers are getting served in this regime with HP overload, the results of Proposition 5.2 are relevant as they continue to hold in two cases where some LP customers *are* getting served: (1) in the fluid limit for the practically important case with non-stationary arrivals and alternating HP under/overload (see §6.6, Proposition 7), and (2) in stochastic systems with heavy HP load but $\rho_1 < 1$.

Proposition 5.2 delivers two main insights on the queue length effects of information:

(1) Considering each priority class in isolation, the effects of information and the underlying driver, are fully consistent with the single-class results of Proposition 2: For each class, no information (N) yields a longer (shorter) queue than full information (F), if this class' position parameter β is below (above) a threshold. For example, for the LP class, Parts 2(a) and (b) establish that, holding β_1 constant, N yields a longer LP queue than F if $\beta_2 < \beta_2^*(\beta_1)$, and vice versa if $\beta_2 > \beta_2^*(\beta_1)$.

(2) Considering both priority classes *jointly*, the information design may have *opposite* effects on their queue lengths: Compared to full information, no information increases the queue of the

relatively optimistic class (β below the class threshold) and decreases the queue of the relatively pessimistic class (β above the class threshold). For example (see Part 2(b)), for $\beta_1 < \beta_1^*(\beta_2)$ and $\beta_2 > \beta_2^*(\beta_1)$, no information increases the HP queue and reduces the LP queue.

Example 2. To illustrate Proposition 5.2, Figure 5 shows how the ranking of the equilibrium numbers-in-system under no and full information depends on $\boldsymbol{\beta}$ at low ($\rho_2 = 0.1$) and high ($\rho_2 = 1$) LP loads. Each point in Figure 5 corresponds to a (β_1, β_2) combination. Figure 5 shows:

1. *Information effect on HP queue length mainly depends on β_1: Figures 5(a)-(b).*

Figures 5(a)-(b) show that, regardless of the LP load, the HP queue length is minimized under full information for low β_1, and under no information for high β_1.

2. *Information effect on LP queue length also depends on LP load ρ_2: Figures 5(c)-(d).*

Figure 5(c) shows that for low LP load ($\rho_2 = 0.1$), the threshold $\beta_2^*(\beta_1) = 1$ for $\beta_1 > 0.75$; this means no information yields a *longer* LP queue ($\bar{x}_2^N(\boldsymbol{\beta}) > \bar{x}_2^F$), *even if* LP customers are maximally pessimistic, i.e., $\beta_2 = 1$. In contrast, Figure 5(d) shows that for high LP load ($\rho_2 = 1$), the threshold $\beta_2^*(\beta_1) < 1$ for all β_1, meaning that no information yields a *shorter* LP queue ($\bar{x}_2^N(\boldsymbol{\beta}) < \bar{x}_2^F$), if LP customers are sufficiently pessimistic, i.e., $\beta_2 \in (\beta_2^*(\beta_1), 1]$.

3. *Consistent vs. opposite information effects on HP and LP queue lengths: Figures 5(e)-(f).*

The blue and yellow areas in Figures 5(e)-(f) correspond to cases where information design has consistent effects on the queue lengths of both classes: The blue area corresponds to Part 2(a), i.e., both β_1 and β_2 are below the respective thresholds, so full information minimizes the queue lengths of *both* classes. The yellow area corresponds to Part 2(d), i.e., both β_1 and β_2 exceed the respective thresholds, so no information minimizes the queue lengths of *both* classes.

In contrast, the green and red areas in Figures 5(e)-(f) correspond to cases where information design is subject to a trade-off between high- and low-priority queue lengths, i.e., one design minimizes the HP queue length but the other minimizes the LP queue length: The green area corresponds to Part 2(b), i.e., β_1 is below and β_2 above the respective threshold, so that full information minimizes the HP queue length but no information minimizes the LP queue length. Similarly, the red area corresponds to Part 2(c), i.e., β_1 is above and β_2 below the respective threshold, so no information minimizes the HP queue but full information minimizes the LP queue.

6.6. Two-Class Priority System with Non-Stationary Periodic Arrivals

We turn to the general case with two priority classes and non-stationary arrivals. This case gives rise to three possible HP load regimes:

(i) Uniformly underloaded HP class ($\bar{\rho}_1 \leq 1$). In this load regime, information only affects the LP class, and the results are consistent with those for single-class non-stationary systems (Propositions 3 and 4). We provide analytical results for this regime in Proposition 6. These results also extend

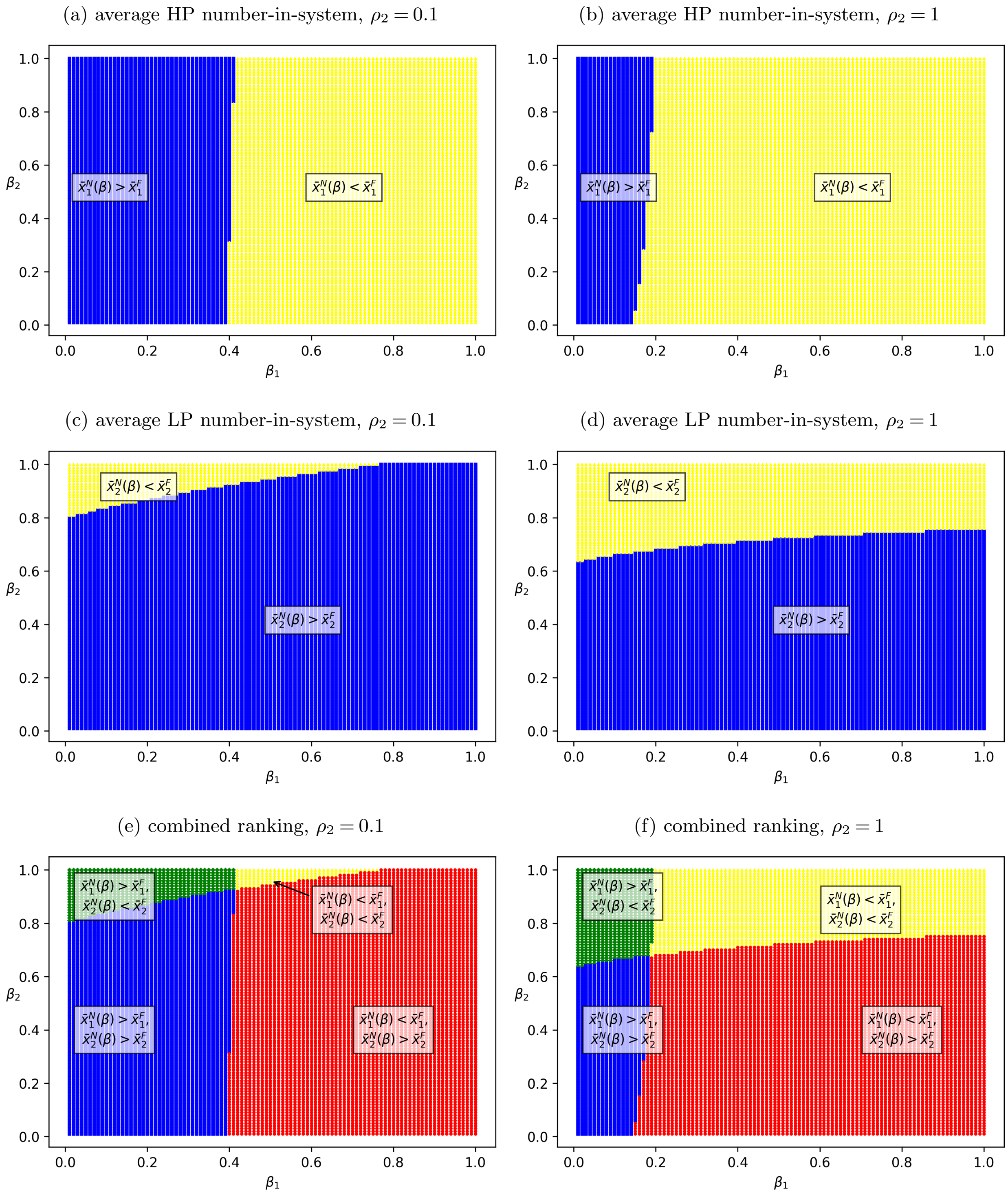

Figure 5 **Stationary two-priority system: Ranking of equilibrium numbers-in-system under information regimes N and F, as function of β ($\mu_1 = \mu_2 = 1,\ s = 100, \theta(x) = 2 - e^{-x}, \rho_1 = 1.5$).**

the single-class results to systems with time-varying capacity.

(ii) HP class alternating between under- and overloaded ($\underline{\rho}_1 < 1 < \bar{\rho}_1$). This practically prevalent load regime shows that the results for simpler systems in §6.3-6.5 are robust, and also gives rise to an important additional trade-off, between HP and LP abandonment, that does not arise in simpler systems. This is technically the most challenging regime. We present a combination of analytical results (Propositions 7 and 8) and numerical results (Example 3 and Figure 6).

(iii) Uniformly overloaded HP class ($\bar{\rho}_1 \geq 1$). In this load regime, information only affects the number-in-system, and the respective ranking results are consistent with those for two-class stationary systems (Proposition 5.2). For the sake of brevity, we omit these results and related discussion.

Uniformly Underloaded HP class ($\bar{\rho}_1 \leq 1$): Proposition 6 summarizes how information affects the equilibrium average numbers-in-system and average abandonment rates.

Proposition 6. *For two-priority systems with non-stationary periodic arrivals and uniformly underloaded HP class (i.e., $\bar{\rho}_1 \leq 1$), the equilibrium average numbers-in-system and abandonment rates under no (N) and full (F) information compare as follows:*

1. *Numbers-in-system: $\bar{x}_1^N(\boldsymbol{\beta}) = \bar{x}_1^F \leq s$. For LP, if $\max_{t\geq 0}(\tilde{x}_1^N(t,(\beta_1,0)) + \tilde{x}_2^N(t,(\beta_1,0))) > s$, there is a threshold $\beta_q^* \in (0, 0.5)$ such that $\bar{x}_2^N(\beta_1, \beta_q^*) = \bar{x}_2^F$ and:*
   (a) *If $\beta_2 \in [0, \beta_q^*)$, then $\bar{x}_2^N(\boldsymbol{\beta}) > \bar{x}_2^F$.*
   (b) *If $\beta_2 \in (\beta_q^*, 1]$, then $\bar{x}_2^N(\boldsymbol{\beta}) < \bar{x}_2^F$.*
2. *Abandonment: $\bar{A}_1^N(\boldsymbol{\beta}) = \bar{A}_1^F = 0$. For LP, if $\max_{t\geq 0}(\tilde{x}_1^N(t,(\beta_1,0)) + \tilde{x}_2^N(t,(\beta_1,0))) > s > \min_{t\geq 0}(\tilde{x}_1^N(t,(\beta_1,1)) + \tilde{x}_2^N(t,(\beta_1,1)))$, there is a threshold $\beta_a^* \in (0, 0.5)$ such that $\bar{A}_2^N(\beta_1, \beta_a^*) = \bar{A}_2^F$ and:*
   (a) *If $\beta_2 \in [0, \beta_a^*)$, then $\bar{A}_2^N(\boldsymbol{\beta}) < \bar{A}_2^F$.*
   (b) *If $\beta_2 \in (\beta_a^*, 1]$, then $\bar{A}_2^N(\boldsymbol{\beta}) > \bar{A}_2^F$.*

We note that the values of the thresholds β_q^* and β_a^* are close but need not coincide.

In systems with uniform HP underload, information clearly has no effect on HP customers because they never queue. Therefore, for LP customers the system is equivalent to a single-class non-stationary system with time-varying capacity. Proposition 6 shows that our results on the performance effects of information generalize naturally from the single-class non-stationary case with constant capacity (Propositions 3.2 and 4.2) to the LP class in two-class systems with uniform HP underload. Specifically, for sufficiently small β_2, no (N) information yields a longer LP queue length (at all times) and lower average LP abandonment rate, compared to full (F) information. Conversely, for sufficiently large β_2, no (N) information yields a shorter LP queue length (at all times) and higher average LP abandonment rate, compared to full (F) information (unless LP customers never queue even under no information, i.e., $\max_{t\geq 0}(\tilde{x}_1^N(t,\boldsymbol{\beta}) + \tilde{x}_2^N(t,\boldsymbol{\beta})) \leq s$). These results on the trade-off between less queueing and more abandonment under no vs. full information are consistent with those for single-class systems with non-stationary arrivals with constant capacity (Propositions 3.2 and 4.2) and also extend these results to systems with time-varying capacity.

HP Class Alternating Between Under- And Overloaded ($\underline{\rho}_1 < 1 < \bar{\rho}_1$): We turn to the practically prevalent case with alternating HP under-/overload. This is also the technically most challenging case. We present a combination of analytical (Propositions 7 and 8) and numerical results (Example 3 and Figure 6). For one, we show that our results for simpler systems are robust, specifically, the two performance trade-offs: (i) between number-in-system and abandonment of the same class under non-stationary arrivals (Propositions 3 and 4 for single-class systems, Proposition 6 for LP class in two-class systems with uniform HP underload), and (ii) between the HP and LP numbers-in-system (Proposition 5.2 for stationary arrivals). Furthermore, we identify an *important additional trade-off, between HP and LP abandonment*, which is unique to this load regime.

Proposition 7 establishes the following information effects on the equilibrium numbers-in-system.

PROPOSITION 7. *For two-priority systems with non-stationary periodic arrivals and at least occasional HP overload ($\bar{\rho}_1 > 1$), the equilibrium numbers-in-system under no (N) and full (F) information compare as follows:*

1. *If $\beta_1 = 0$, then for HP: $\tilde{x}_1^N(t,\boldsymbol{\beta}) \geq \tilde{x}_1^F(t)\ \forall t$, and $\bar{x}_1^N(\boldsymbol{\beta}) > \bar{x}_1^F$ if $\max_{t\geq 0} \tilde{x}_1^N(t,\boldsymbol{\beta}) > s$.*
   For LP customers:
   (a) *$\beta_2 = 0$: Then $\tilde{x}_2^F(t) < \tilde{x}_2^N(t,\boldsymbol{\beta})\ \forall t$, if $\min_{t\geq 0} \tilde{x}_1^F(t) \geq s$.*
   (b) *$\beta_2 = 1$: Then $\tilde{x}_2^F(t) > \tilde{x}_2^N(t,\boldsymbol{\beta})\ \forall t$, if $\max_{t\geq 0} \tilde{x}_1^F(t) > s$ and $\mu_2 \leq \theta(0)$.*
2. *If $\beta_1 \geq 0.5$ then for HP: $\tilde{x}_1^N(t,\boldsymbol{\beta}) \leq \tilde{x}_1^F(t)\ \forall t$, and $\bar{x}_1^N(\boldsymbol{\beta}) < \bar{x}_1^F$ if $\max_{t\geq 0} \tilde{x}_1^F(t) > s$.*
   For LP customers:
   (a) *$\beta_2 = 0$: Then $\tilde{x}_2^F(t) < \tilde{x}_2^N(t,\boldsymbol{\beta})\ \forall t$, if $\max_{t\geq 0} \tilde{x}_1^F(t) > s$ and $\mu_2 \leq \theta(0)$.*
   (b) *$\beta_2 = 1$: Then there are LP load thresholds tilde$\rho_2^1 < \tilde{\rho}_2^2$ such that*
      i. *$\tilde{x}_2^F(t) < \tilde{x}_2^N(t,\boldsymbol{\beta})\ \forall t$, if $\bar{\rho}_2 < \tilde{\rho}_2^1$ and $\min_{t\geq 0} \tilde{x}_1^N(t,\boldsymbol{\beta}) \geq s$.*
      ii. *$\tilde{x}_2^F(t) > \tilde{x}_2^N(t,\boldsymbol{\beta})\ \forall t$, if $\underline{\rho}_2 > \tilde{\rho}_2^2$ and $\min_{t\geq 0} \tilde{x}_1^N(t,\boldsymbol{\beta}) \geq s$, or $\max_{t\geq 0} \tilde{x}_1^N(t,\boldsymbol{\beta}) > s$ and $\mu_2 \geq \theta(\infty)$.*

REMARK 3. Proposition 7 provides results on the *uniform ranking* of the equilibrium processes at *all* times t, focusing on four combinations of low and high β_1 and β_2, i.e., $\beta_1 \in \{0, [0.5, 1]\}$ and $\beta_2 \in \{0, 1\}$. These results imply the same rankings for the time *averages* of these processes, and also highlight how the ranking of these time averages depends more generally on the $\boldsymbol{\beta}$ parameters: Specifically, the ranking of average queue lengths for class i under F vs. N reverses as the queue position perception parameter β_i increases from low to high. Our numerical results illustrate this structure for the entire $\boldsymbol{\beta}$ range (Example 3 and Figure 6).

The key insights from Proposition 7 on the queue length effects of information are consistent with those of Proposition 5.2 for the stationary two-class case with HP overload:

(1) For each priority class, compared to full information (F), no information (N) generally yields a longer queue if the class position parameter β is low, and a longer queue if β is high.

(2) The information design may have *opposite* effects on the queue lengths of the two classes. Specifically, compared to full information (F), no information (N) generally increases the queue of the optimistic class ($\beta = 0$) and decreases the queue of the pessimistic class ($\beta = 1$). Furthermore, information may also have opposite queue length effects if both classes are relatively pessimistic and the LP load is below a treshold: Specifically, by Part 2(b)i. of Proposition 7, for $\beta_1 \geq 0.5$ and $\beta_2 = 1$, full information increases the HP queue length, but reduces the LP queue length if the LP load is below a threshold ($\bar{\rho}_2 < \tilde{\rho}_2^1$). This is consistent with the stationary case; see Figure 5(e).

Proposition 8 establishes the following ranking of the *average* abandonment rates.

PROPOSITION 8. *For two-priority systems with non-stationary periodic arrivals and HP class that alternates between under- and overloaded ($\underline{\rho}_1 < 1 < \bar{\rho}_1$), information has the following effects on the average abandonment rates:*

1. *If $\beta_1 = 0$, then for HP: $\bar{A}_1^N(\boldsymbol{\beta}) \leq \bar{A}_1^F$, with strict inequality iff $\max_{t\geq 0} \tilde{x}_1^N(t, \boldsymbol{\beta}) > s > \min_{t\geq 0} \tilde{x}_1^F(t)$.*
   For LP customers:
   (a) $\beta_2 = 0$: *Then $\bar{A}_2^N(\boldsymbol{\beta}) = \bar{A}_2^F$ if $\min_{t\geq 0} \tilde{x}_1^F(t) \geq s$.*
   (b) $\beta_2 = 1$: *Then $\bar{A}_2^N(\boldsymbol{\beta}) > \bar{A}_2^F$, if $\max_{t\geq 0} \tilde{x}_1^F(t) > s > \min_{t\geq 0} \tilde{x}_1^F(t)$ and $\mu_2 \leq \theta(0)$.*
2. *If $\beta_1 \geq 0.5$ then for HP: $\bar{A}_1^N(\boldsymbol{\beta}) \geq \bar{A}_1^F$, with strict inequality iff $\max_{t\geq 0} \tilde{x}_1^F(t) > s > \min_{t\geq 0} \tilde{x}_1^N(t, \boldsymbol{\beta})$.*
   For LP customers:
   (a) $\beta_2 = 0$: *Then $\bar{A}_2^N(\boldsymbol{\beta}) < \bar{A}_2^F$, if $\max_{t\geq 0} \tilde{x}_1^F(t) > s > \min_{t\geq 0} \tilde{x}_1^N(t, \boldsymbol{\beta})$ and $\mu_2 \leq \theta(0)$.*
   (b) $\beta_2 = 1$: *Then $\bar{A}_2^N(\boldsymbol{\beta}) = \bar{A}_2^F$ if $\min_{t\geq 0} \tilde{x}_1^N(t, \boldsymbol{\beta}) \geq s$.*

Propositions 7 and 8 show that information design involves two abandonment-related trade-offs in non-stationary two-priority systems with alternating HP over-/underload:

(1) Between abandonment and number-in-system for each class. This trade-off is consistent with settings where a single class experiences queueing (Propositions 3.2, 4.2, and 6).

(2) Between the HP and LP abandonment rates. This trade-off arises only in non-stationary systems with two classes that experience queueing. Specifically, if $\beta_1 = 0$ and $\beta_2 = 1$, no information (N) minimizes the HP abandonment rate whereas full information (F) minimizes the LP abandonment rate (see Part 1(b)), and vice versa if $\beta_1 \geq 0.5$ and $\beta_2 = 0$ (see Part 2(a)).

Numerical Study. We conclude the analysis of non-stationary two-class priority systems with a numerical study that shows how the theoretical ranking results of Propositions 7 and 8 for specific low/high (β_1, β_2) pairs extend to the entire $\boldsymbol{\beta}$ range. We focus on the time-averages of the

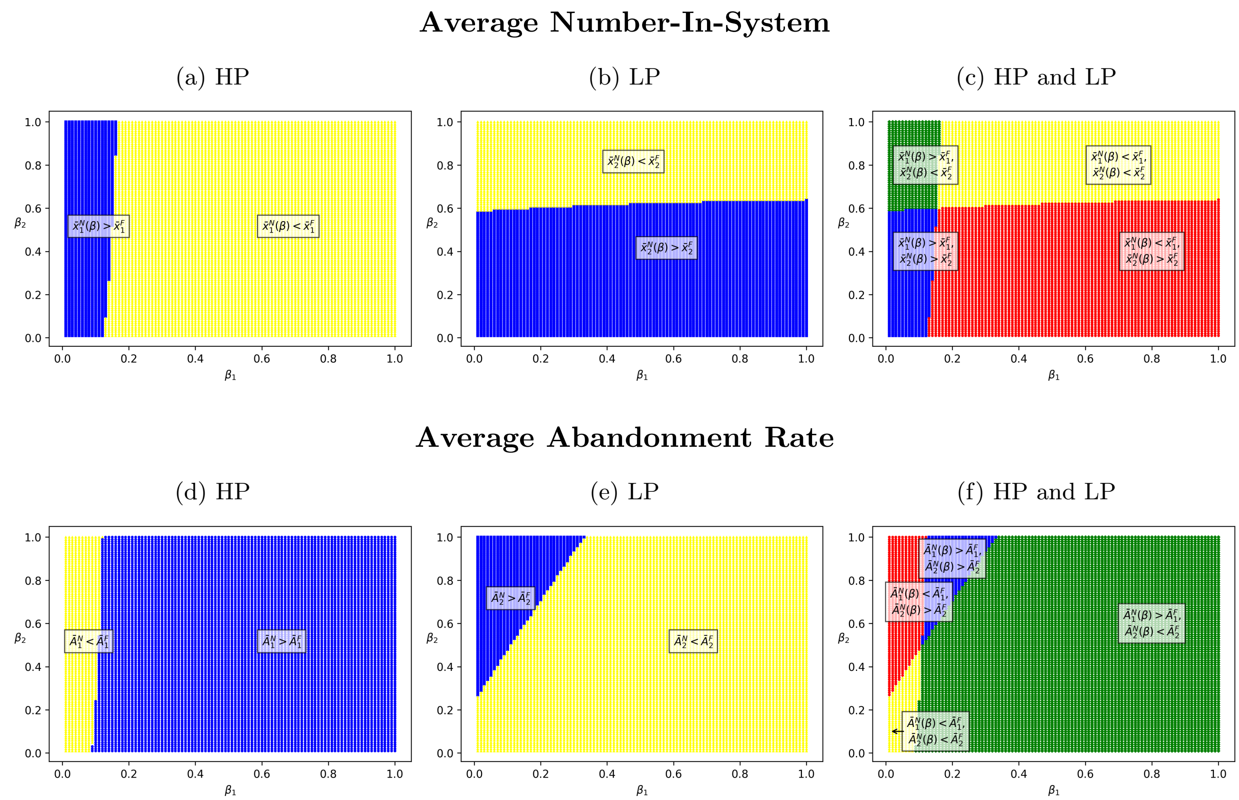

Figure 6 Non-stationary two-priority system: Ranking of equilibrium performance metrics under information regimes N and F, as function of β

$\mu_1 = \mu_2 = 1,\ s = 100,\ \rho_1 = 0.9,\ \rho_2 = 0.5,\ \ \lambda_k(t) = s\rho_k(1 - 0.5\sin(\pi t/12)),\ \theta(x) = 2 - e^{-x}$.

equilibrium numbers-in-system and abandonment rates, because (i) these averages are of first-order importance, and (ii) the time-varying processes need not obey a uniform ranking at all times.

Example 3. We consider the same supply parameters ($\mu_1 = \mu_2 = 1$, $s = 100$) and abandonment rate function ($\theta(x) = 2 - e^{-x}$) as in Example 2, but the following more moderate and time-varying arrival rates: $\lambda_k(t) = s\rho_k(1 - 0.5\sin(\pi t/12))$, for $k = 1, 2$, where $\rho_1 = 0.9$ and $\rho_2 = 0.5$ are the average HP and LP loads, respectively. Note that $\theta(0) = \mu_2 < \theta(\infty)$, $\underline{\rho}_1 = 0.45 < 1$, and $\bar{\rho}_1 = 1.35 > 1$. These parameters satisfy the conditions in Parts 1(b) and 2(a) of Propositions 7 and 8. Figure 6 shows the rankings of the equilibrium average numbers-in-system (plots (a)-(c)) and average abandonment rates (plots (d)-(f)) under no (N) vs. full (F) information, as functions of $(\beta_1, \beta_2) \in [0, 1] \times [0, 1]$.

Figure 6 shows the following information design effects, consistent with Propositions 7 and 8:

1. *Trade-off between HP number-in-system and abandonment rate:* Figures 6(a) and (d) show that, if HP customers are sufficiently optimistic (e.g. $\beta_1 < 0.1$), then full information (F) reduces the number-in-system but increases the abandonment rate, compared to no information (N); if HP customers are sufficiently pessimistic (e.g., $\beta_1 > 0.2$) then full information has the opposite effects.

2. *Trade-off or alignment between LP number-in-system and abandonment rate:* Figures 6(b) and (e) show that for LP customers, information design involves a similar trade-off between queueing and abandonment, as for HP customers, with the following qualification: For $\beta_1 \leq 0.4$, this trade-off arises for sufficiently low or high β_2. However, for $\beta_1 > 0.4$, this trade-off arises only if LP customers are sufficiently optimistic ($\beta_2 \lessapprox 0.6$), whereas no information minimizes *both*, number-in-system and abandonment, if LP customers are more pessimistic ($\beta_2 \gtrapprox 0.6$).

3. *Consistent vs. opposite information effects on HP and LP numbers-in-system:* Figure 6(c) shows that the information design may have consistent or opposite effects on the two classes. The blue and yellow areas correspond to $\boldsymbol{\beta}$ parameters with consistent information effects for both classes: Both queue lengths are minimized under full information, if both classes are sufficiently optimistic (blue area), and under no information if they are sufficiently pessimistic (yellow area).

In contrast, the red and green areas correspond to $\boldsymbol{\beta}$ parameters that yield opposite information effects on the HP and LP classes. The red area corresponds to $\boldsymbol{\beta}$ parameters (sufficiently pessimistic HP and optimistic LP customers) where full information increases the HP queue but reduces the LP queue. The green area corresponds to $\boldsymbol{\beta}$ parameters (sufficiently optimistic HP and pessimistic LP customers) where no information yields the same trade-off, longer HP but shorter LP queue. These results are also consistent with the stationary case; see Proposition 5.2 and Figure 5(f).

4. *Consistent vs. opposite information effects on HP and LP abandonment rates:* Figure 6(f) similarly shows that the information design may have consistent effects (blue and yellow areas) or opposite effects (red and green areas) on the abandonment rates of the two classes, depending on customers' queue position perceptions in the absence of full information.

7. Robustness Checks

In this section, we establish the robustness of our key results if one relaxes important assumptions. In Section 7.1, we consider a generalized no-information model with waiting-time dependent queue position perceptions; we establish the robustness of our comparison results under this generalized model. In Section 7.2, we show that our comparison results based on fluid approximations are also valid for small and moderately-sized systems. In Section 7.3 we explain how our performance comparison results apply to the waiting time metric. Finally, in Section 7.4 we discuss how our performance comparison results apply to settings with time-varying capacity.

7.1. A Generalized Model of Waiting-Time-Dependent Queue Position Perception

Our no-information model captures class-k customers' perceived queue position via the class-dependent *constant* position fraction β_k. A more general no-information model would allow the position fraction to be dynamic and waiting-time-dependent. Denote by $w_{kl}(t)$ the elapsed waiting time of the class k customer in position l of her class by time t and $\boldsymbol{w}(t)$ as the vector containing $w_{kl}(t)$ for all customers in queue. Then, a more general no-information model would assume a dynamic and waiting-time-dependent position fraction, i.e., $\beta(w_{kl}(t))$. For simplicity, we refer to the no-information model with such waiting-time-dependent position fractions as the $\beta(\boldsymbol{w}(t))$ *model*. Recall that we call the model with class-dependent constant β that we analyzed so far the β_k *model*. In this section, we show numerically that this class-dependent β_k model serves as an accurate approximation to the $\beta(\boldsymbol{w}(t))$ model. Before we proceed with details, we summarize the model assumptions, key finding, and interpretations.

- *Model*. We assume that $\beta : \mathbb{R}_0^+ \to (0,1]$ is a non-increasing function with $\beta(0) = 1$ and $lim_{x\to\infty}\beta(x) = 0$. These assumptions reflect that customers perceive to be at the end of the queue upon arrival ($\beta(0) = 1$) and progressively improve their perception (lower their β) as their waiting time accumulates. We explore a wide range of non-increasing $\beta(x)$ functions; see Table 1. Throughout, we assume a common $\beta(x)$ function for both classes, mainly for the sake of simplicity.
- *Key finding*. For both stationary and non-stationary two-class priority systems, the equilibrium average numbers-in-system and abandonment rates for any given $\beta(x)$ function, can be accurately approximated by appropriately choosing the (β_1, β_2) pair in our β_k model.
- *Interpretations*.

(i) It is intuitive that our β_k model can be "tuned" to match the performance under the more general $\beta(\boldsymbol{w}(t))$ model: Whereas the $\beta(\boldsymbol{w}(t))$ model tracks the waiting-time-dependent, and therefore heterogeneous, queue position beliefs of all present customers at all times, the appropriately chosen β_k parameters simply reflect the class-level averages of these beliefs over time. Therefore, the values of the matching (β_1, β_2) depend on both the $\beta(x)$ function's rate of decline, and the waiting times experienced in each class.
(ii) For a *common* (class-independent) $\beta(x)$ function (our focus in this section), we consistently observe $\beta_1 > \beta_2$, as HP customers have shorter waiting times than LP customers; see Table 2.
(iii) However, we think our β_k model should be similarly accurate in approximating the performance for a *class-dependent* $\beta(\boldsymbol{w}(t))$ model.[3] Such a class-dependent model is flexible enough to capture the notion that LP customers may be somehow aware of their lower priority - even without being explicitly informed, and may therefore perceive their queue positions to improve at a slower rate, so that $\beta_1'(x) < \beta_2'(x) < 0$. In that case, the values of the matching β_k model would reflect on average more optimistic HP customers, i.e., $\beta_1 < \beta_2$.

Analytical approach. The theoretical analysis of the performance of the waiting-time-dependent $\beta(\boldsymbol{w}(t))$ model is challenging due to $\boldsymbol{w}(t)$ being a vector with time-varying, state-dependent, and continuous components $w_{kl}(t)$. Therefore, obtaining the equilibrium of the system

[3] This also requires matching the same number of performance metrics using two parameters, as for a common $\beta(x)$.

by establishing a convergence theorem for this model, and conducting further performance analysis, are difficult. However, we can gain insights into its performance by simulating the corresponding stochastic system. Therefore, we assess the flexibility of our class-dependent β_k model by comparing the two key performance metrics, average numbers-in-system and abandonment rates, derived from the simulated stochastic system under the $\beta(\boldsymbol{w}(t))$ model against those obtained from the fluid-based β_k model. We do so for two primary reasons. First, comparing with fluid β_k model approximations, rather than the actual stochastic measures, is computationally efficient and reduces the computational error. Second, the primary focus of this paper is on comparing performance metrics in fluid equilibrium under different information models. Therefore, we aim to demonstrate that the β_k model serves as a good approximation of the $\beta(\boldsymbol{w}(t))$ model under fluid scaling.

Assumptions for $\beta(\boldsymbol{w}(t))$ model. We assume that $\beta(w_{kl}(t))$ is a value in $(0,1]$ representing a customer's belief about her relative position in queue after waiting for $w_{kl}(t)$ units of time. A lower $\beta(w_{kl}(t))$ value signifies a belief in being closer to the front of the queue. Therefore, when $w_{kl}(t)$ is small, a newly arrived customer perceives herself to be at the tail of the queue, indicated by $\beta(w_{kl}(t))$ approaching 1. As the waiting time $w_{kl}(t)$ accumulates, the customer anticipates progressing forward in the queue, resulting in a decrease in $\beta(w_{kl}(t))$. Conversely, when $w_{kl}(t)$ is large, the customer assumes a position closer to the head of the queue, with $\beta(w_{kl}(t))$ approaching 0. Thus, it is natural to assume that $\beta(w_{kl}(t))$ is non-increasing in $w_{kl}(t)$, as we do here.

Simulation experiments. We use simulation to examine the robustness of our approximation. Assume that $\mu_1 = 1, \mu_2 = 1,\ s = 100$. Given the absence of empirical evidence on the form of $\beta(x)$, we explore a wide range of non-increasing $\beta(x)$ functions, including both convex to concave shapes. These are detailed in the first column of Table 1 and plotted in Figure 7. We also explore various abandonment rate functions $\theta(x)$; see the second column of Table 1.

We investigate both stationary and non-stationary arrivals, each with different sets of average system loads, as shown in the last column of Table 1. Specifically, when the arrival rates are non-stationary, we examine the following set of sinusoidal arrival rates with the average system loads provided in Table 1: $\lambda_1(t) = \bar{\rho}_1\mu_1 s(1-0.5\sin(\pi t/12))$, $\lambda_2(t) = \bar{\rho}_2\mu_2 s(1-0.5\sin(\pi t/12))$.

For each set of parameters and specific form of $\beta(\cdot)$, we first generate the sample path of the number-in-system process $X^N(t,\beta(\boldsymbol{w}(t)))$ under waiting-time-dependent $\beta(\boldsymbol{w}(t))$ by generating the arrival, service, and abandonment processes of the system. Note that, the aggregate class k abandonment rate of the abandonment process at time u is defined as:

$$\mathcal{A}_1(X^N(u,\beta(\boldsymbol{w}(u)))) := \sum_{l=1}^{(X_1^N(u,\beta(\boldsymbol{w}(u)))-s^n)^+} \theta\left(\frac{\beta(w_{1l}(u))(X_1^N(u,\beta(\boldsymbol{w}(u))) + X_2^N(u,\beta(\boldsymbol{w}(u)) - s)^+}{s}\right),$$

$$\mathcal{A}_2(X^N(t,\beta(\boldsymbol{w}(t)))) := \sum_{l=1}^{\left(X_2^N(u,\beta(\boldsymbol{w}(u))-(s-X_1^N(u,\beta(\boldsymbol{w}(u))))^+\right)^+} \theta\left(\frac{\beta(w_{2l}(u))(X_1^N(u,\beta(\boldsymbol{w}(u))) + X_2^N(u,\beta(\boldsymbol{w}(u)) - s)^+}{s}\right).$$

$\beta(x)$		$\theta(x)$	$(\bar{\rho}_1, \bar{\rho}_2)$
reciprocal	$\beta(x) = 1/(1 + cx)$	$2 - e^{-x}$	(1.5, 0.5)
exp	$\beta(x) = e^{-cx}$	$2 - e^{-0.5x}$	(2, 0.5)
linear	$\beta(x) = (1 - x/r)^+$	$2 - e^{-2x}$	(1, 0.5)
constant	$\beta(x) = 1 - i/r$, for $i \le x < i+1, i = 0, \ldots, r-1$.	$1.5 - e^{-x}$	(1.5, 1)
concave1	$\beta(x) = ((1 - x/r)^{1/m})^+$	$2 - 0.5 * e^{-x}$	(1.5, 1.5)
concave2	$\beta(x) = (1 - (x/r)^m)^+$		

Table 1 **Parameter sets explored for model validation in Section 7.1.**

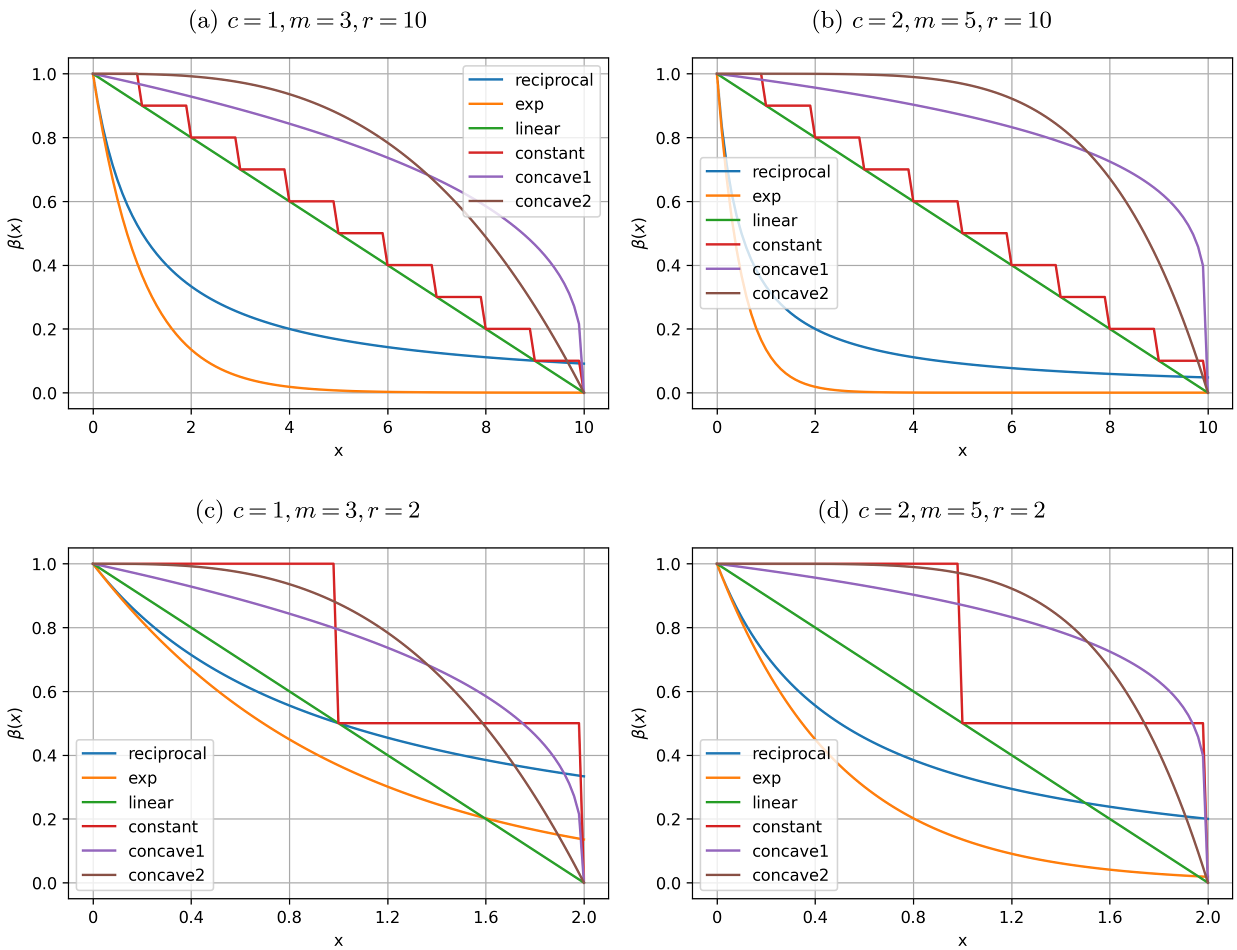

Figure 7 **Different forms of $\beta(x)$ with various sets of parameters (refer to the function of $\beta(x)$ in Table 1)**

Ideally, we should continuously update the abandonment rates for customers waiting in queue since $\boldsymbol{w}(t)$ evolves continuously. However, due to the complexity of simulating a nonhomogeneous Poisson process with an endogenously changing rate function, we instead update the individual abandonment rates for all customers in queue whenever there is a change in the system state. Given the large scale of our simulated system (with $s = 100$), the system state updates approximately every 0.005 units of time. This ensures that the differences between our simulated abandonment processes and the actual abandonment processes remain small.

We generate the sample path of the $\beta(\boldsymbol{w}(t))$ model starting from an empty system, utilizing the initial $t=100$ as the warm-up period. By allowing the simulation to run for a sufficient duration (up to time $t=1100$), we consistently observe convergence to an equilibrium. Consequently, based on the system state updates after the warm-up period, we compute the performance measures for this system, including the long-run average numbers-in-system $(\bar{x}^N_{1,w}, \bar{x}^N_{2,w})$, the long-run average system abandonment rates $(\bar{A}^N_{1,w}, \bar{A}^N_{2,w})$ and, in cases of non-stationary and periodic arrival rates, the time-varying average numbers-in-system over a period $(\tilde{x}^N_{1,w}(t), \tilde{x}^N_{2,w}(t))$. For each parameter set, we compare the performance measures obtained from the simulated stochastic system under the $\beta(\boldsymbol{w}(t))$ model with those obtained from the fluid equilibrium under the β_k model.

Stationary arrivals. When arrival rates are stationary, we observe that, for each set of parameters, the simulated sample path of the $\beta(\boldsymbol{w}(t))$ model converges to an equilibrium point $(\bar{x}^N_{1,w}, \bar{x}^N_{2,w})$. To identify the β_k model that best approximates the $\beta(\boldsymbol{w}(t))$ model, we match the average numbers-in-system as follows: (1) We substitute $(\bar{x}^N_1, \bar{x}^N_2) = (\bar{x}^N_{1,w}, \bar{x}^N_{2,w})$ into equations (39) and (41) to determine β^*_1 such that these two equations hold. (2) We substitute $(\bar{x}^N_1, \bar{x}^N_2) = (\bar{x}^N_{1,w}, \bar{x}^N_{2,w})$ into equations (40) and (43) to determine β^*_2 such that these two equations hold. The β_k model with $\boldsymbol{\beta} = \boldsymbol{\beta}^* := (\beta^*_1, \beta^*_2)$ yields the desired approximation of the $\beta(\boldsymbol{w}(t))$ model. We observe that such $\boldsymbol{\beta}^*$ always exists, and the resulting β_k model yields approximately the same average abandonment rates (relative errors less than 2%) as the $\beta(\boldsymbol{w}(t))$ model.

Let $\mu_1 = 1$, $\mu_2 = 1$, $s = 100$, $\theta(x) = 2 - e^{-x}$, $\rho_1 = 1.5$, $\rho_2 = 0.5$, and $\beta(x) = (1 - \frac{x}{2})^+$. In Figure 8, the blue lines illustrate the trajectories of the averages from 50 simulated sample paths of the numbers-in-system processes, truncated after running for 200 time units, under the $\beta(\boldsymbol{w}(t))$ model (with 95% confidence interval provided). The red lines represent the equilibrium numbers-in-system under the approximating β_k model with $\beta^*_1 = 0.9$ and $\beta^*_2 = 0.59$.[4] As illustrated in Figure 8, the β_k model serves as a very good approximation of the $\beta(\boldsymbol{w}(t))$ model.

Non-stationary arrivals. We now consider the case when arrival rates are non-stationary and sinusoidal. Similar to our approach in the stationary case, we identify the approximating β_k model by matching the long-run average numbers-in-system with those of the $\beta(\boldsymbol{w}(t))$ model. In the non-stationary and periodic case, the equilibrium of the system under the fluid β_k model is time-varying and periodic. Therefore, we can not determine the optimal $\boldsymbol{\beta}^*$ by directly solving the balance equations as we did in the stationary case. For each long-run average number-in-system $(\bar{x}^N_{1,w}, \bar{x}^N_{2,w})$ obtained from the simulated stochastic system under $\beta(\boldsymbol{w}(t))$ model, we find the optimal

[4] As explained above, that $\beta^*_1 > \beta^*_2$ follows because our numerical study restricts attention to a common $\beta(x)$ function.

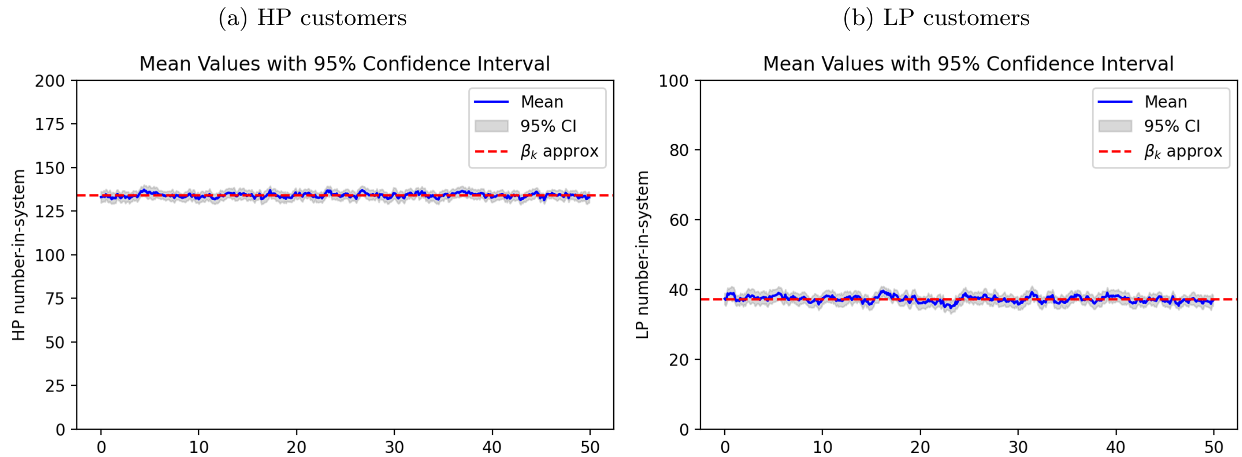

Figure 8 **Average of 50 sample paths of the numbers-in-system under $\beta(\boldsymbol{w}(t))$ model (with 95% confidence interval provided) and the corresponding optimal β_k model ($\mu_1 = 1, \mu_2 = 1$, $s = 100, \theta(x) = 2 - e^{-x}, \rho_1 = 1.5, \rho_2 = 0.5$, $\beta(x) = (1 - \frac{x}{2})^+$).**

$\boldsymbol{\beta}^*$ for β_k model as follows. For each pair of (β_1, β_2) with $\beta_k \in \{0, 0.01, \dots, 1\}, k \in \{1, 2\}$, we calculate the time-average numbers-in-system of the fluid equilibrium under the β_k model, denoted by $(\bar{x}_1^N(\boldsymbol{\beta}), \bar{x}_2^N(\boldsymbol{\beta}))$. We then select (β_1, β_2) such that the average relative gap between $(\bar{x}_1^N(\boldsymbol{\beta}), \bar{x}_2^N(\boldsymbol{\beta}))$ and $(\bar{x}_{1,w}^N, \bar{x}_{2,w}^N)$, defined as $MAPE_x(\boldsymbol{\beta}) := (|\bar{x}_1^N(\boldsymbol{\beta}) - \bar{x}_{1,w}^N| / \bar{x}_1^N(\boldsymbol{\beta}) + |\bar{x}_2^N(\boldsymbol{\beta}) - \bar{x}_{2,w}^N| / \bar{x}_{2,w}^N)/2$, is minimized. The identified value of (β_1, β_2) is denoted as $\boldsymbol{\beta}^* = (\beta_1^*, \beta_2^*)$. As illustrated in Table 2, the β_k model with $\boldsymbol{\beta} = \boldsymbol{\beta}^*$ yields average numbers-in-system very close to those of the $\beta(\boldsymbol{w}(t))$ model, with average relative gaps ($MAPE_x(\boldsymbol{\beta}^*)$) of less than 0.2%.

In the non-stationary case, we not only focus on aligning the long-run average numbers-in-system but also check whether the periodic equilibria are matched between the two models. Therefore, we compare the time-varying periodic average trajectories of the number-in-system processes under the $\beta(\boldsymbol{w}(t))$ model (i.e., $(\tilde{x}_{1,w}^N(t), \tilde{x}_{2,w}^N(t))$) and the corresponding β_k model with $\boldsymbol{\beta}^*$ (i.e., $(\tilde{x}_1^N(t, \boldsymbol{\beta}^*), \tilde{x}_2^N(t, \boldsymbol{\beta}^*))$). As illustrated in Figure 9, the trajectories of the HP number-in-system overlap, indicating that our optimal β_k model with $\boldsymbol{\beta}^*$ serves as a reliable approximation for the $\beta(\boldsymbol{w}(t))$ model, even in the context of the time-varying HP number-in-system. For the LP class, the equilibrium fluid trajectory under the β_k model exhibits a comparable yet smoother shape, with a reduced amplitude, compared to the $\beta(\boldsymbol{w}(t))$ model.

Now that we have illustrated the robustness of the approximating β_k model in terms of the average number-in-system, we turn to examining whether this model serves as a reliable approximation for the average system abandonment rate. We do so by evaluating the average relative gap of the long-run time-average system abandonment rates, defined as $MAPE_a(\boldsymbol{\beta}) := (|\bar{A}_1^N(\boldsymbol{\beta}) - \bar{A}_{1,w}^N| / \bar{A}_1^N(\boldsymbol{\beta}) + |\bar{A}_2^N(\boldsymbol{\beta}) - \bar{A}_{2,w}^N| / \bar{A}_{2,w}^N)/2$, between the $\beta(\boldsymbol{w}(t))$ model and the optimal β_k model. Table 2 presents the comparative results between these two models in terms of the time-average

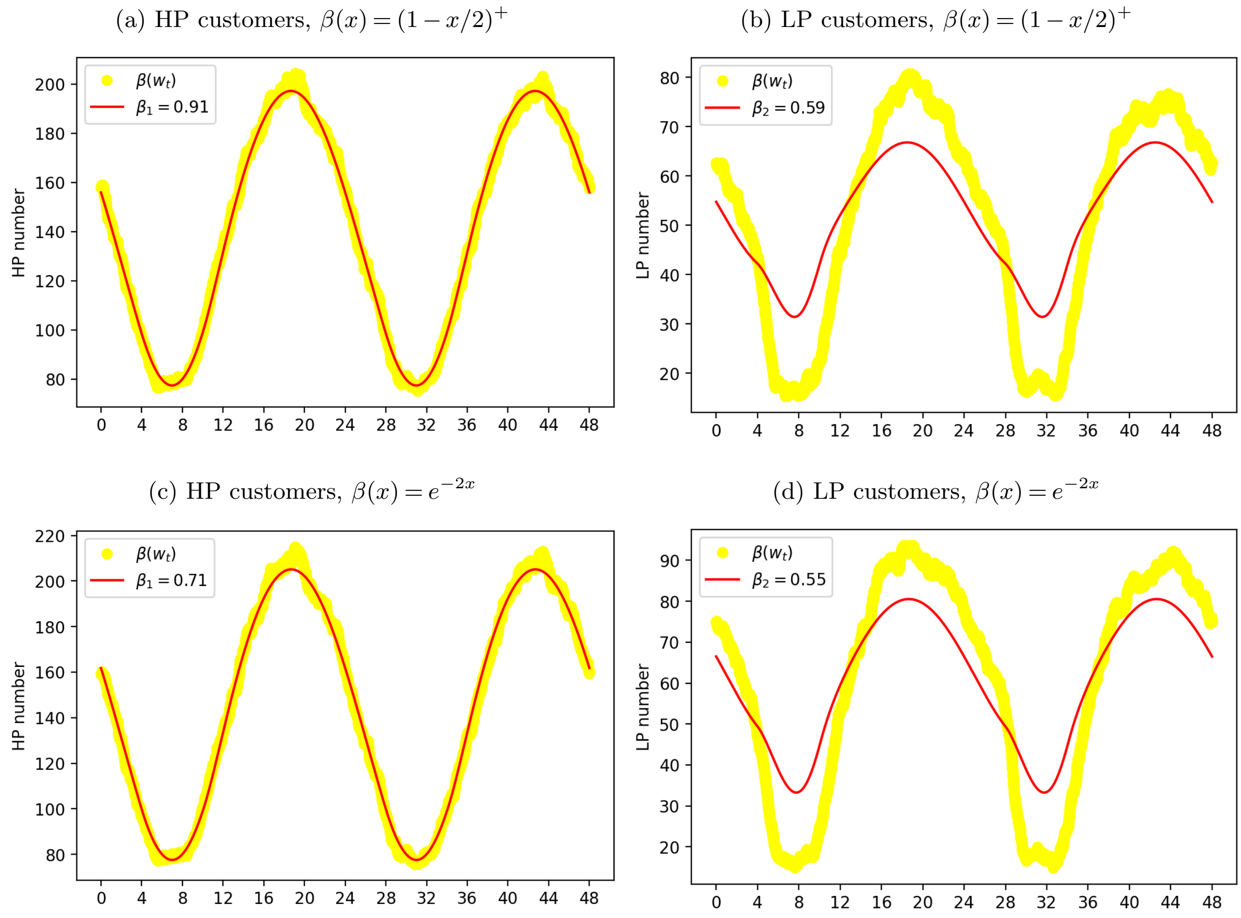

Figure 9 Trajectories of numbers-in-system processes in periodic equilibrium under the optimal β_k models, compared with the average sample path of the $\beta(\boldsymbol{w}(t))$ model ($\mu_1 = 1, \mu_2 = 1,$ $s = 100, \theta(x) = 1.5 - e^{-x}, \lambda_1(t) = 150(1 - 0.5\sin(\pi t/12)), \lambda_2(t) = 50(1 - 0.5\sin(\pi t/12))$).

$\beta(x)$	$\beta(\boldsymbol{w}(t))$ model				optimal β_k model					$MAPE_x(\boldsymbol{\beta}^*)$	$MAPE_a(\boldsymbol{\beta}^*)$
	$\bar{x}^N_{1,w}$	$\bar{x}^N_{2,w}$	$\bar{A}^N_{1,w}$	$\bar{A}^N_{2,w}$	$\boldsymbol{\beta}^*$	$\bar{x}^N_1(\boldsymbol{\beta}^*)$	$\bar{x}^N_2(\boldsymbol{\beta}^*)$	$\bar{A}^N_1(\boldsymbol{\beta}^*)$	$\bar{A}^N_2(\boldsymbol{\beta}^*)$		
exp	144.86	60.72	54.64	42.62	(0.61,0.30)	144.90	60.84	53.77	46.23	0.11%	5.02%
reciprocal	144.41	51.46	54.59	42.79	(0.69, 0.59)	144.45	51.37	53.77	46.23	0.10%	4.76%
linear	140.94	51.78	54.65	42.86	(0.91, 0.59)	140.95	51.82	53.83	46.17	0.04%	4.62%
constant	140.33	47.47	54.64	42.94	(0.99, 0.84)	140.51	47.48	53.83	46.17	0.08%	4.50%
concave1	140.69	46.24	54.64	43.08	(0.99, 0.94)	140.66	46.19	53.83	46.17	0.06%	4.33%
concave2	140.30	46.13	54.62	42.99	(0.99, 0.94)	140.66	46.19	53.83	46.17	0.19%	4.42%

Table 2 Performance comparisons between $\beta(\boldsymbol{w}(t))$ model and optimal β_k model for various $\beta(x)$ functions. ($\mu_1 = 1, \mu_2 = 1,$ $s = 100, \theta(x) = 1.5 - e^{-x}, \lambda_1(t) = 150(1 - 0.5\sin(\pi t/12)), \lambda_2(t) = 50(1 - 0.5\sin(\pi t/12)), c = 2, m = 5, r = 2.$)

system abandonment rates for various $\beta(x)$ functions. We observe that the approximating β_k model also maintains a small gap in the time-average system abandonment rates (less than 5%).

We also find that our comparative results between no and full information designs are consistent under the β_k and $\beta(\boldsymbol{w}(t))$ models, despite the small difference in average abandonment rates.

$\beta(x)$	HP abandonment rate			LP abandonment rate		
	No info with $\beta(\boldsymbol{w}(t))$	No info with optimal β_k	Full info	No info with $\beta(\boldsymbol{w}(t))$	No info with optimal β_k	Full info
exp	54.64	53.77	53.50	42.62	46.23	46.50
reciprocal	54.59	53.77	53.50	42.79	46.23	46.50
linear	54.65	53.83	53.50	42.86	46.17	46.50
constant	54.64	53.83	53.50	42.94	46.17	46.50
concave1	54.64	53.83	53.50	43.08	46.17	46.50
concave2	54.62	53.83	53.50	42.99	46.17	46.50

Table 3 Comparison of average abandonment rates between no information $\beta(\boldsymbol{w}(t))$ model, no information optimal β_k model, and full information model for various $\beta(x)$ functions. ($\mu_1 = 1, \mu_2 = 1$, $s = 100, \theta(x) = 1.5 - e^{-x}, \lambda_1(t) = 150(1 - 0.5\sin(\pi t/12)), \lambda_2(t) = 50(1 - 0.5\sin(\pi t/12)), c = 2, m = 5, r = 2.$

We illustrate this using the same sets of parameter values as in Table 2. We obtain the average abandonment rates under full information system, $\bar{A}_1^F, \bar{A}_2^F$, and compare them with those obtained from the no information models in Table 3. As shown in Table 3, our comparisons between the no information model with β_k and full information models regarding the average abandonment rates remain robust.

Note that the impact of different $\beta(x)$ functions on the long-run average system abandonment rates is limited. This is because the variation in the form of $\beta(x)$ function is inherently bounded by 1 (by definition) and further constrained by the range of the $\theta(x)$ function. Additionally, in highly overloaded systems (as in Table 3), servers are almost always busy regardless of the $\beta(x)$ function used, resulting in a similar level of throughput across all systems. Since the difference in system abandonment rates corresponds to the difference in system throughputs, these differences should be small. On the other hand, when system loads are low, abandonment rates are low, and $\beta(x)$ has low impact on the system through abandonments.

7.2. Small Stochastic Systems

In Theorem 1, we establish the convergence of fluid-scaled stochastic processes to corresponding fluid limits, as the system size increases without bound. In this section, we use simulation experiments to illustrate the accuracy of the fluid approximations as well as our comparison results for systems with a moderate number of servers (e.g., less than 50).

We consider two-class systems with time-varying arrival rates, under no and full information models. Specifically, we assume that $\mu_1 = \mu_2 = 1$, $s = 20$, $\theta(x) = 5 - 4e^{-x}$, $\boldsymbol{\beta} = (1, 1)$, and $\lambda_k(t) = \rho_k \mu_k s(1 - 0.8\sin(\pi t/12))$. We consider two sets of system load when the system switches between over- and under-loaded: relatively low load with $\rho_1 = \rho_2 = 0.8$, and high load with $\rho_1 = \rho_2 = 1.5$. For each information level, we estimate the expected average number of customers in periodic steady-state using simulation. In Figures 10 and 11, we plot 95% confidence intervals for the expected number-in-system process, under each information level, along with corresponding time-dependent fluid limits, at equilibrium, over one period, for $\rho_1 = \rho_2 = 0.8$ and 1.5, respectively. Based on Figures 10 and 11, we make two observations:

(a) HP customers (b) LP customers

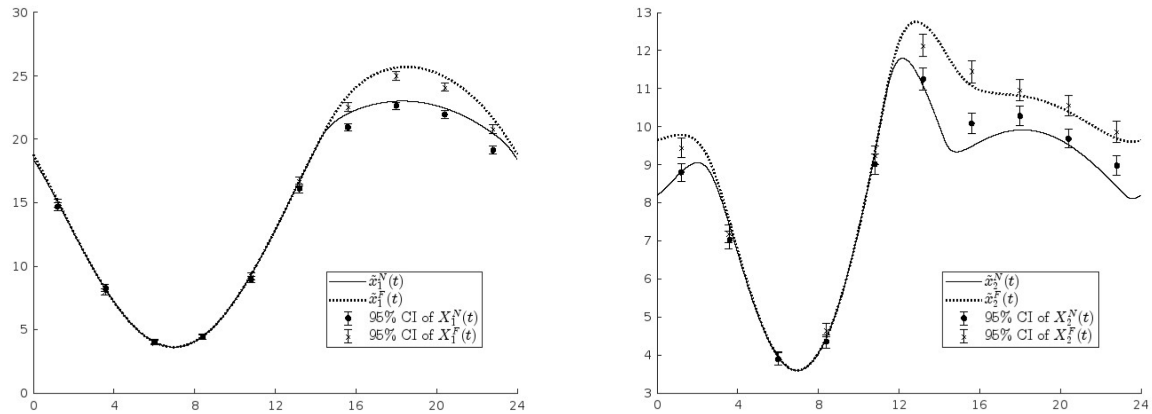

Figure 10 Comparisons of the number-in-system trajectories under different information levels for the simulated stochastic systems and the fluid models ($\mu_1 = \mu_2 = 1$, $s = 20, \theta(x) = 5 - 4e^{-x}, \boldsymbol{\beta} = (1,1), \rho_1 = \rho_2 = 0.8, \lambda_k(t) = \rho_k \mu_k s(1 - 0.8\sin(\pi t/12))$**).**

(a) HP customers (b) LP customers

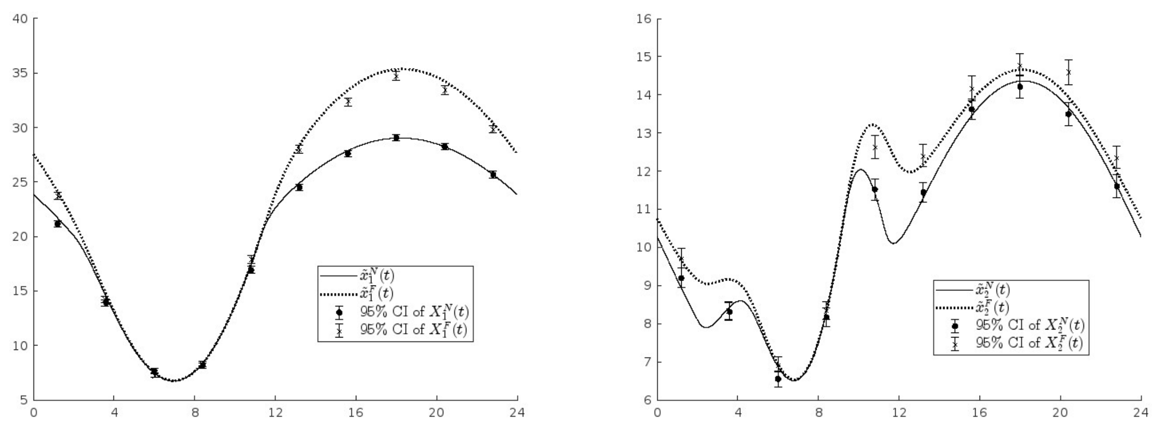

Figure 11 Comparisons of the number-in-system trajectories under different information levels for the simulated stochastic systems and the fluid models ($\mu_1 = \mu_2 = 1$, $s = 20, \theta(x) = 5 - 4e^{-x}, \boldsymbol{\beta} = (1,1), \rho_1 = \rho_2 = 1.5, \lambda_k(t) = \rho_k \mu_k s(1 - 0.8\sin(\pi t/12))$**).**

1. For each information level, the periodic fluid equilibrium curve is very close to the corresponding simulation-based estimate of the expected number-in-system, which implies that our fluid approximations are fairly accurate even with a small (i.e., $s = 20$) number of servers.

2. The simulation-based ranking is consistent with the fluid-based ranking at each point in time. In particular, the number-in-system rankings in our numerical examples lie in Cases 2 of Proposition 7 and align with Figure 6.(c).

For further robustness checks, we consider systems with alternative sizes, e.g., $s = 10$ or 50; our observations and conclusions are consistent with those for the case of $s = 20$; see Appendix D.

Next, we examine the accuracy of the fluid-based average system abandonment rankings for the stochastic systems. In particular, for various sets of parameters, we estimate the expected average system abandonment rates of the stochastic systems under no and full information and obtain their rankings (statistically significant at 95% confidence level) using simulation. We observe that, when $s = 20$, the simulation-based average system abandonment rate rankings are consistent with the fluid-based rankings.

Overall, the results indicate that our fluid-based average number-in-system and abandonment rankings are also valid for small and moderately-sized stochastic systems.

7.3. Comparing information levels with respect to waiting time

Our results focus on the two key performance metrics of queue length and abandonment rate. Another important performance metric is the average waiting time. This metric is connected to the queue length through Little's Law and, as such, the comparative results in Section 6 concerning long-run average performance also apply to the waiting time metric. Specifically, by Theorem LL.2 in John (2011) we have that, under information design I, $L_k^I = \lambda_k W_k^I$, where L_k^I is the equilibrium class k average queue length, λ_k is the class k average arrival rate, and W_k^I is the equilibrium class k average waiting time. Note that, by the remarks on Theorem LL.1 in John (2011), Little's law holds irrespective of the queue discipline and under non-stationary arrivals, as long as we are concerned with long-run average performance; see also Theorem 2.1. of Whitt (2015). Therefore, we can deduce the waiting time rankings from the average queue length rankings presented in Section 6.

7.4. Non-stationary number of servers

Our paper focuses on systems with a static number of servers, s. In practice, however, the number of servers may be non-stationary and vary over the course of a week or a day to accommodate service providers' preferences or constraints. For cases where the number of servers is non-stationary and periodic, denoted as $s(t)$, our main results generalize as illustrated below.

For the single-class case, analyzing a system with a non-stationary number of servers $s(t)$ is equivalent to focusing on LP customers in a system with two classes, non-stationary arrivals, and a uniformly underloaded HP class, where the comparison results are provided in Proposition 6. This implies that our comparison results for single-class system with a stationary number of servers generalize to the system with a non-stationary number of servers.

For the two-class case, denote $\bar{s} := \max_{t \geq 0} s(t)$ and $\Delta s(t) = \bar{s} - s(t)$, then a two-class system with non-stationary number of servers $s(t)$ and HP arrival rate $\lambda_1(t)$ is equivalent to a system with a stationary number of servers $\bar{s}$ and HP arrival rate $\lambda_1(t) + \mu_1 \Delta s(t)$. As such, our results are also relevant for systems with non-stationary number of servers.

8. Conclusions

8.1. Summary of Main Results

This paper contributes theoretical models and insights on the effects of providing information in observable service systems with abandonment. We consider a Markovian queueing system with two priority classes (HP and LP), time-varying arrival rates and abandonment, a setting that is practically relevant but has hardly been studied in the information design literature. We propose a fairly flexible model of how information impacts customer abandonment, which captures key empirical findings in the literature pertaining to customer abandonment from observable queues.

Our results characterize the effects of information on key performance metrics of abandonment and number-in-system (or waiting time) and provide insights on how these effects depend on the interplay between: (i) Customers' perceived queue position under no information (parameters β_1 and β_2); (ii) class-specific system load; (iii) temporal variability of arrival rates, and (iv) priority service. In the presence of time-varying arrivals and with two priority classes, we observe the following key effects and trade-offs:

- Number-in-system (and waiting time): HP-LP trade-off. The information design has *opposite* effects on the queue lengths of the HP and LP classes, if their customers have sufficiently different queue position perceptions. In such cases, compared to full information, no information increases the queue of the relatively optimistic class (β below a threshold) and decreases the queue of the relatively pessimistic class (β above a threshold). For example, for sufficiently low β_1 and high β_2, no information increases the HP queue and reduces the LP queue, compared to full information. (See Proposition 5 for overloaded stationary systems, and Proposition 7 and Figure 3.(c) for systems with alternating HP under-/overload.)
- Abandonment: HP-LP trade-off in systems with non-stationary HP under-/overload. Two-priority non-stationary systems give rise to an additional trade-off, between HP and LP abandonment, if the HP class alternates between under- and overloaded. In such cases, compared to full information, no information reduces the abandonment of the relatively optimistic class (β below a threshold) and increases the abandonment of the relatively pessimistic class (β above a threshold). For example, for sufficiently low β_1 and high β_2, no information reduces the HP abandonment and increases the LP abandonment, compared to full information. (See Proposition 8 and Figure 3.(f).)

8.2. Managerial Implications

Our results imply that effective information provision requires (i) first identifying for a particular system whether the above trade-offs indeed exist, given the load conditions and customer perceptions (β parameters). (ii) In cases where such trade-offs do exist, information design must carefully balance the queueing and abandonment costs of both classes, and/or consider hybrid designs (e.g., giving full information to one class but no information to the other class). Operational measures, such as the load, should be readily measurable, whereas customer perceptions, may be estimated using customer surveys. For example, the National Health Service (NHS) in the United Kingdom routinely conducts patient surveys to gather feedback on various aspects of their healthcare experience, including waiting times [5].

As a concrete example, consider the following parameter regime relevant to an ED: HP class alternates between under- and overloaded regimes; HP customers are sufficiently optimistic (i.e., they perceive a low queue position) and LP customers are sufficiently pessimistic (i.e., they perceive a high queue position). In this case, providing accurate information minimizes LP abandonment but has the appositive impact on HP abandonment. Interestingly, the field study of Westphal et al. (2022) found that providing both operational and time information improved the sense of making progress in the ED for patients, but information only reduced abandonment when only operational information (and not waiting time information) was provided. Our results suggest that the impact of information on abandonment depend on both system load and customer perceptions. As such, these factors should be accounted for in design and empirical analysis of future field studies.

Our results also provide insights on how other operational system decisions can impact the effects of information. More specifically, the system manager can (partially) control the system load through staffing (or capacity allocation) decisions. Our results indicate that in scenarios with minimal time-variation in arrival patterns, managing the system load becomes the primary factor determining whether information provision impacts a specific class. Conversely, if arrival patterns do vary over time, the impact of system load management becomes more nuanced, as the effects of information are intertwined with the aforementioned trade-offs. In both scenarios, the system manager can eliminate the information-related performance trade-offs between the two priority classes by setting staffing levels to ensure that the HP class remains underloaded. Specifically, the potentially significant trade-off between the HP and LP abandonment rates in non-stationary settings under alternating HP under-/overload (Proposition 8) vanishes under uniform HP underload (Proposition 6).

[5] See: https://nhssurveys.org/

8.3. Future Research

Our study motivates several future directions. In the following, we briefly discuss a few.

Our results highlight the challenges of effective information design in a complex service setting such as the ED. In particular, the trade-offs that we identify motivate the design of more sophisticated state-dependent and/or hybrid information provision systems. One way to address the queue length and abandonment trade-off when arrivals are time-varying is to explore time/state-dependent information designs. For systems with a priority service, a specific information design may have opposite effects on customers of different classes. Therefore, it would be interesting to study schemes that provide different information levels for different priority classes.

Another approach to tackle the queue length and abandonment trade-off is to consider a socially optimal information design, which appropriately balances the conflicting effects of information on different performance measures and priority classes. This however requires quantifying the relative impact of waiting time and abandonment for different priority classes.

Our characterizations of the effects of information depend on customers' perceptions of their queue positions in the absence of accurate information. These results also suggest that "correcting" customer perceptions, instead of providing accurate queue information about their queue positions, may be sufficient for reducing abandonment. Vague delay announcements have have been previously studied in the context of unobserved (virtual) call center queues, e.g., Allon et al. (2011), Allon and Bassamboo (2011). Investigating the impact of vague announcements in observable settings, aimed at influencing customer perceptions, can be an interesting area of future research.

Our results assume that customers in both classes use the same θ function (that maps their perceived queue positions to individual abandonment rates). As noted in §3, this assumption is not only for analytical tractability, but also allows us to isolate the interactions between information granularity and system characteristics, i.e., non-stationary arrival rates and priority service. To accommodate heterogeneous abandonment behaviors across different classes, one can relax this assumption. Whether customers in different classes respond differently to their perceived queue positions, and if so, how, is ultimately an empirical question, and it is certainly worthy of future research (see below). With respect to the theoretical implications of a model with class-dependent θ functions, it seems intuitively clear that our results would continue to hold if these functions are either "not too different", or if they differ in a way that reinforces the effects that our analysis identifies; on the other hand, some of our results may be reversed if the θ functions differ in a way that counters these effects. For instance, as discussed in §6.5 and §6.6, the effect of more information on the average LP queue length is the net result of two effects (i) HP congestion (more information yields a longer HP queue) and (ii) LP patience (more information weakly reduces LP customers' individual abandonment rates for given queue lengths). In this context, if the HP customers have a "steeper" θ function than LP customers (e.g., $\theta'_1 > \theta'_2$), then this would magnify the HP congestion effect compared to the LP patience effect; as result, the information level that minimizes the LP queue may change for certain load regimes.

In addition to modeling extensions, another interesting direction is to investigate the estimation of the models introduced in this paper from data. Our modeling framework assumes that customers' abandonment rate is determined through a function (namely, θ) that maps the perceived position of the customer to an abandonment rate. Estimation of this function using data, which can be facilitated by imposing additional structure on the function, can be considered in future work. As noted above, customer perceptions may be estimated through customer experience surveys. Such estimations can provide insights on the impact of information on abandonment rates and its heterogeneity between different customer classes.

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Appendix A: Proof of Section 4: Theorem 1

In this section, we provide the proof of Theorem 1. We first introduce a useful Functional Strong Law of Large Numbers from Mandelbaum et al. (1998) in Appendix A.1, and then provide the proof of Theorem 1 for no and full information systems in Appendices A.2 and A.3, respectively.

A.1. Functional Strong Law of Large Numbers

Consider a sequence of stochastic processes $\mathbf{Q}^n := \{\mathbf{Q}^n(t)|t \geq 0\}$ for $n > 0$. The sample paths of $\mathbf{Q}^n$ are uniquely determined by $\mathbf{Q}^n(0)$ and the functional equations

$$\mathbf{Q}^n(t) = \mathbf{Q}^n(0) + \sum_{i \in I} K_i \left(\int_0^t \alpha_s^n(\frac{1}{n}\mathbf{Q}^n(s), i) ds \right) \mathbf{v}_i,$$

where $\{K_i(\cdot)|i \in I\}$ is a collection of mutually independent standard Poisson processes indexed by a countable or countably infinite set I, and are independent of $\mathbf{Q}^n(0)$, $\{\mathbf{v}_i \in \mathbb{V}|i \in I\}$ for a separable Banach space $\mathbb{V}$ with norm $|\cdot|$ is a collection of vectors such that

$$\sum_{i \in I} |\mathbf{v}_i| < \infty, \tag{22}$$

and $\{\alpha_t^n(\cdot, i)|t \geq 0, i \in I\}$ is a collection of real-valued non-negative Lipschitz function on $\mathbb{V}$ that jointly satisfy

$$\|\alpha_t^n(\cdot, i)\| \leq n\beta_t\gamma^{(i)} \tag{23}$$

for some locally integrable function β_t and $\gamma^{(i)} \in \mathbb{R}$, $i \in I$. Note that $\|\cdot\|$ is the **Lipschitz norm** for real-valued functions on $\mathbb{V}$, i.e.,

$$\|f\| := \sup_{x, y \in \mathbb{V}, x \neq y} \frac{|f(x) - f(y)|}{|x - y|} \vee |f(0)|.$$

THEOREM 3. ***Theorem 2.2 in Mandelbaum et al. (1998)*** *Assume that (22) and (23) hold. Moreover, assume that*

$$\lim_{n \to \infty} \sum_{i \in I} \int_0^t \left\| \frac{\alpha_s^n(\cdot, i)}{n} - \alpha_s(\cdot, i) \right\| ds = 0, \tag{24}$$

for all $t \geq 0$, where $\{\alpha_s(\cdot, i)|t \geq 0, i \in I\}$ is a collection of Lipschitz function. If $\{\mathbf{Q}^n(0)|n > 0\}$ is any family of random initial state vectors in $\mathbb{V}$, then

$$\lim_{n \to \infty} \frac{\mathbf{Q}^n(0)}{n} = \mathbf{Q}(0) \ \ a.s. \ \ implies \ \lim_{n \to \infty} \frac{\mathbf{Q}^n(t)}{n} = \mathbf{Q}(t) \ \ a.s.,$$

where the convergence is uniform on compact sets in t, and $\mathbf{Q}$ is the unique deterministic process $\{\mathbf{Q}(t)|t \geq 0\}$ that solves the integral equation

$$\mathbf{Q}(t) = \mathbf{Q}(0) + \int_0^t \sum_{i \in I} \alpha_s(\mathbf{Q}(s), i) \mathbf{v}_i ds, t \geq 0.$$

A.2. Proof of Theorem 1: No Information

We justify the fluid approximation by applying Theorem 3 to the scaled process under no information: $\{X^{N,n}(t)/n : t \geq 0\}$. Let $\mathbb{V} = \mathbb{R}^2$, $I = 1, \dots, 6$, $K_k = A_k, K_{k+2} = S_k, K_{k+4} = N_k$ for $k = 1, 2$, $\mathbf{v}_1 = (1, 0)', \mathbf{v}_2 = (0, 1)', \mathbf{v}_3 = \mathbf{v}_5 = (-1, 0)', \mathbf{v}_4 = \mathbf{v}_6 = (0, -1)'$. Then, (22) holds obviously. Moreover, for $x = (x_1, x_2) \in \mathbb{R}^2$, let

$$\alpha_t(x,1) = \lambda_1(t), \qquad \alpha_t^n(x,1) = \lambda_1^n(t);$$

$$\alpha_t(x,2) = \lambda_2(t), \qquad \alpha_t^n(x,2) = \lambda_2^n(t);$$

$$\alpha_t(x,3) = \mu_1(x_1 \wedge s), \qquad \alpha_t^n(x,3) = \mu_1(nx_1 \wedge s^n);$$

$$\alpha_t(x,4) = \mu_2\left((s - x_1)^+ \wedge x_2\right), \qquad \alpha_t^n(x,4) = \mu_2\left((s^n - nx_1)^+ \wedge nx_2\right);$$

$$\alpha_t(x,5) = \theta\left(\frac{\beta_1(x_1 + x_2 - s)^+}{s}\right)(x_1 - s)^+, \qquad \alpha_t^n(x,5) = \theta\left(\frac{\beta_1(nx_1 + nx_2 - s^n)^+}{s^n}\right)(nx_1 - s^n)^+;$$

$$\alpha_t(x,6) = \theta\left(\frac{\beta_2(x_1 + x_2 - s)^+}{s}\right)(x_2 - (s - x_1)^+)^+, \quad \alpha_t^n(x,6) = \theta\left(\frac{\beta_2(nx_1 + nx_2 - s^n)^+}{s^n}\right)(nx_2 - (s^n - nx_1)^+)^+.$$

Then, we need to verify the assumptions of 3, i.e., (23)–(24), and $\alpha_t(x,i)$ being Lipschitz for $i = 1, \dots, 6$.

First, we show that $\alpha_t(x,i)$ is Lipschitz, i.e., $\|\alpha_t(x,i)\| < \infty$, for $i = 1, \dots, 6$. Since $\alpha_t(x,1)$ and $\alpha_t(x,2)$ are independent of x, the proof is trivial. Recall that $|\cdot|$ is the standard Euclidean norm on $\mathbb{R}^2$, then

$$\|\alpha_t(x,3)\| = \sup_{x,y \in \mathbb{R}^2} \frac{|\mu_1(y_1 \wedge s) - \mu_1(x_1 \wedge s)|}{|y - x|} \leq \sup_{x,y \in \mathbb{R}^2} \frac{\mu_1|y_1 - x_1|}{|y - x|} \leq \sup_{x,y \in \mathbb{R}^2} \frac{\mu_1|y_1 - x_1|}{|y_1 - x_1|} = \mu_1.$$

$$\|\alpha_t(x,4)\| = \sup_{x,y \in \mathbb{R}^2} \frac{\mu_2|(s - y_1)^+ \wedge y_2 - (s - x_1)^+ \wedge x_2|}{|y - x|} =: \sup_{x,y \in \mathbb{R}^2} L_1(x,y).$$

1. If $x_1, y_1 > s$, $L_1(x,y) = 0$.
2. If $x_1 \leq s, y_1 > s$, then
$$L_1(x,y) = \frac{\mu_2|(s - x_1) \wedge x_2|}{|y - x|} \leq \frac{\mu_2|(y_1 - x_1) \wedge x_2|}{|y - x|} \leq \frac{\mu_2|(y_1 - x_1)|}{|y_1 - x_1|} = \mu_2.$$
3. If $x_1 > s, y_1 \leq s$, similar to the previous case, we can obtain that $L_1(x,y) \leq \mu_2$.
4. If $x_1, y_1 \leq s$, then
$$\begin{aligned} L_1(x,y) &= \frac{\mu_2|(s - y_1) \wedge y_2 - (s - x_1) \wedge x_2|}{|y - x|} \\ &= \begin{cases} \frac{\mu_2|(s-y_1)\wedge y_2 - x_2|}{|y-x|} \leq \frac{\mu_2|y_2 - x_2|}{|y_2 - x_2|} = \mu_2, & \text{if } x_1 + x_2 \leq s; \\ \frac{\mu_2|(s-y_1)\wedge y_2 - (s-x_1)|}{|y-x|} \leq \frac{\mu_2|(s-y_1)-(s-x_1)|}{|y_1 - x_1|} = \mu_2, & \text{if } x_1 + x_2 > s. \end{cases} \end{aligned}$$

Thus, $\|\alpha_t(x,4)\| \leq \mu_2 < \infty$.

$$\|\alpha_t(x,5)\| = \sup_{x,y \in \mathbb{R}^2} \frac{\left|\theta\left(\frac{\beta_1(y_1+y_2-s)^+}{s}\right)(y_1 - s)^+ - \theta\left(\frac{\beta_1(x_1+x_2-s)^+}{s}\right)(x_1 - s)^+\right|}{|y - x|} =: \sup_{x,y \in \mathbb{R}^2} L_2(x,y).$$

Note that, $\theta'(x)x \leq \theta(x)$, for $x > 0$, since θ is increasing and concave. Therefore, $\theta'(x)x \leq M$. For $x, y \in \mathbb{R}^2$, without loss of generality, assume that $x_1 + x_2 \leq y_1 + y_2$, then

1. If $x_1, y_1 > s$,
$$\begin{aligned} L_2(x,y) &= \frac{\left|\theta\left(\beta_1 \frac{y_1+y_2-s}{s}\right)(y_1 - s) - \theta\left(\beta_1 \frac{x_1+x_2-s}{s}\right)(x_1 - s)\right|}{|y - x|} \\ &\leq \frac{\left|\theta\left(\beta_1 \frac{y_1+y_2-s}{s}\right)(y_1 - x_1)\right|}{|y - x|} + \frac{\left|\theta\left(\beta_1 \frac{y_1+y_2-s}{s}\right) - \theta\left(\beta_1 \frac{x_1+x_2-s}{s}\right)\right|(x_1 - s)}{|y - x|} \\ &\leq \frac{|M(y_1 - x_1)|}{|y_1 - x_1|} + \frac{\left|\theta'\left(\beta_1 \frac{x_1+x_2-s}{s}\right) \frac{y_1+y_2-x_1-x_2}{s/\beta_1}\right|(x_1 + x_2 - s)}{|y - x|} \\ &\leq M + M\frac{|y_1 - x_1| + |y_2 - x_2|}{|y - x|} \leq M + \sqrt{2}M. \end{aligned}$$

2. If $x_1 \le s, y_1 > s$,

$$L_2(x,y) = \frac{\left|\theta\left(\beta_1 \frac{y_1+y_2-s}{s}\right)(y_1-s)\right|}{|y-x|} \le \frac{M(y_1-x_1)}{|y-x|} \le M.$$

3. If $x_1 > s, y_1 \le s$, then similarly we can obtain that $L_2(x,y) \le M$.
4. If $x_1, y_1 \le s$, then $L_2(x,y) = 0$.

Therefore, $\|\alpha_t(x,5)\| \le (1+\sqrt{2})M < \infty$.

$$\begin{aligned}\|\alpha_t(x,6)\| &= \sup_{x,y\in\mathbb{R}^2} \frac{\left|\theta\left(\frac{\beta_2(y_1+y_2-s)^+}{s}\right)(y_2-(s-y_1)^+)^+ - \theta\left(\frac{\beta_2(x_1+x_2-s)^+}{s}\right)(x_2-(s-x_1)^+)^+\right|}{|y-x|} \\ &=: \sup_{x,y\in\mathbb{R}^2} L_3(x,y).\end{aligned}$$

For $x, y \in \mathbb{R}^2$, without loss of generality, assume that $x_1 + x_2 \le y_1 + y_2$, then

1. If $x_1, y_1 > s$,

$$\begin{aligned}L_3(x,y) &= \frac{\left|\theta\left(\beta_2 \frac{y_1+y_2-s}{s}\right)y_2 - \theta\left(\beta_2 \frac{x_1+x_2-s}{s}\right)x_2\right|}{|y-x|} \\ &\le \frac{\left|\theta\left(\beta_2 \frac{y_1+y_2-s}{s}\right)(y_2-x_2)\right|}{|y-x|} + \frac{\left|\theta\left(\beta_2 \frac{y_1+y_2-s}{s}\right) - \theta\left(\beta_2 \frac{x_1+x_2-s}{s}\right)\right|x_2}{|y-x|} \\ &\le M + \frac{\left|\theta'\left(\beta_2 \frac{x_1+x_2-s}{s}\right)\frac{y_1+y_2-x_1-x_2}{s/\beta_2}\right|(x_1+x_2-s)}{|y-x|} \\ &\le M + M\frac{|y_1-x_1|+|y_2-x_2|}{|y-x|} \le M + \sqrt{2}M.\end{aligned}$$

2. If $x_1 \le s, y_1 > s$,
   (a) If $x_1 + x_2 \le s$, then

$$L_3(x,y) = \frac{\left|\theta\left(\beta_2 \frac{y_1+y_2-s}{s}\right)y_2\right|}{|y-x|} \le \frac{M|y_1+y_2-x_1-x_2|}{|y-x|} \le \sqrt{2}M.$$

   (b) If $x_1 + x_2 > s$, then

$$\begin{aligned}L_3(x,y) &= \frac{\left|\theta\left(\beta_2 \frac{y_1+y_2-s}{s}\right)y_2 - \theta\left(\beta_2 \frac{x_1+x_2-s}{s}\right)(x_1+x_2-s)\right|}{|y-x|} \\ &\le \frac{\left|\theta\left(\beta_2 \frac{y_1+y_2-s}{s}\right)(y_1+y_2-s) - \theta\left(\beta_2 \frac{x_1+x_2-s}{s}\right)(x_1+x_2-s)\right|}{|y-x|} \\ &\le \frac{\left|\theta\left(\beta_2 \frac{y_1+y_2-s}{s}\right)(y_1+y_2-x_1-x_2)\right|}{|y-x|} + \frac{\left|\theta\left(\beta_2 \frac{y_1+y_2-s}{s}\right) - \theta\left(\beta_2 \frac{x_1+x_2-s}{s}\right)\right|(x_1+x_2-s)}{|y-x|} \\ &\le 2M\frac{|y_1-x_1|+|y_2-x_2|}{|y-x|} \le 2\sqrt{2}M.\end{aligned}$$

3. If $x_1 > s, y_1 \le s$, then $y_1 + y_2 > s$, and

$$\begin{aligned}L_3(x,y) &= \frac{\left|\theta\left(\beta_2 \frac{y_1+y_2-s}{s}\right)(y_1+y_2-s) - \theta\left(\beta_2 \frac{x_1+x_2-s}{s}\right)x_2\right|}{|y-x|} \\ &\le \frac{\left|\theta\left(\beta_2 \frac{y_1+y_2-s}{s}\right)y_2 - \theta\left(\beta_2 \frac{x_1+x_2-s}{s}\right)x_2\right|}{|y-x|} \le \sqrt{2}M.\end{aligned}$$

4. If $x_1, y_1 \le s$,
   (a) If $x_1 + x_2 > s$, then

$$L_3(x,y) = \frac{\left|\theta\left(\beta_2 \frac{y_1+y_2-s}{s}\right)(y_1+y_2-s) - \theta\left(\beta_2 \frac{x_1+x_2-s}{s}\right)(x_1+x_2-s)\right|}{|y-x|} \le 2\sqrt{2}M.$$

(b) If $x_1+x_2\le s, y_1+y_2>s$, then
$$L_3(x,y)=\frac{\left|\theta\left(\beta_2\frac{y_1+y_2-s}{s}\right)(y_1+y_2-s)\right|}{|y-x|}\le M\frac{|y_1+y_2-x_1-x_2|}{|y-x|}\le\sqrt{2}M.$$

(c) If $y_1+y_2\le s$, then $L_3(x,y)=0$.

Thus, $\|\alpha_t(x,6)\|\le 2\sqrt{2}M<\infty$.

Using a similar analysis, we can obtain that $\frac{\alpha_t^n(x,i)}{n}$ is Lipschitz (i.e., $\|\alpha_t^n(x,i)\|\le nM^*$ for some constant $M^*<\infty$) for $i=1,\dots,6$.

Lastly, we prove equation (24). For $i=1,2$, since $\lambda_i^n(t)/n\to\lambda_i(t)$ is bounded, by the dominated convergence theorem we have
$$\lim_{n\to\infty}\int_0^t\left\|\frac{\alpha_u^n(x,i)}{n}-\alpha(x,i)\right\|du=\lim_{n\to\infty}\int_0^t\left\|\frac{\lambda_i^n(t)}{n}-\lambda_i(t)\right\|du=0.$$
For $i=4,\dots,6$, $\alpha_t^n(\cdot,i)$ are independent of t, thus it is sufficient to prove that $\lim_{n\to\infty}\left\|\frac{\alpha_t^n(x,i)}{n}-\alpha(x,i)\right\|=0$. Since $s^n/n\to s$, we have $\lim_{n\to\infty}\left\|\frac{s^n}{n}-s\right\|=0$. Moreover, $\alpha_t^n(x,i)$ and $\alpha(x,i)$ are continuous functions of x with $\lim_{n\to\infty}\frac{\alpha_t^n(x,i)}{n}=\alpha(x,i)$ for $i=4,\dots,6$. Therefore, $\lim_{n\to\infty}\left\|\frac{\alpha_t^n(x,i)}{n}-\alpha(x,i)\right\|=0$, for $i=4,\dots,6$.

Apply Theorem 3 we can obtain the desired results.

A.3. Proof of Theorem 1: Full Information

For full information system, the proof is similar to the no information case except that now
$$\alpha_t(x,5)=\int_0^{(x_1-s)^+}\theta\left(\frac{u}{s}\right)du,\qquad \alpha_t^n(x,5)=\sum_{i=1}^{\lfloor nx_1-s^n\rfloor^+}\theta\left(\frac{i}{s^n}\right);$$
$$\alpha_t(x,6)=\int_{(x_1-s)^+}^{(x_1+x_2-s)^+}\theta\left(\frac{u}{s}\right)du,\quad \alpha_t^n(x,6)=\sum_{i=\lfloor nx_1-s^n\rfloor^++1}^{\lfloor nx_1+nx_2-s^n\rfloor^+}\theta\left(\frac{i}{s^n}\right),$$
where $\lfloor x\rfloor$ is the floor of x. Therefore, to apply Theorem 3, we just need to verify (23)–(24), and $\alpha_t(x,i)$ being Lipschitz for $i=5,6$.

First, we prove that $\alpha_t(x,i)$ is Lipschitz for $i=5,6$. Note that,
$$\begin{aligned}\|\alpha_t(x,5)\|&=\sup_{x,y\in\mathbb{R}^2}\frac{\left|\int_0^{(y_1-s)^+}\theta\left(\frac{u}{s}\right)du-\int_0^{(x_1-s)^+}\theta\left(\frac{u}{s}\right)du\right|}{|y-x|}\\&=\sup_{x,y\in\mathbb{R}^2}\frac{\left|\int_{(x_1-s)^+}^{(y_1-s)^+}\theta\left(\frac{u}{s}\right)du\right|}{|y-x|}\le\sup_{x,y\in\mathbb{R}^2}\frac{M|y_1-x_1|}{|y-x|}\le M<\infty.\end{aligned}$$
$$\begin{aligned}\|\alpha_t(x,6)\|&=\sup_{x,y\in\mathbb{R}^2}\frac{\left|\int_{(y_1-s)^+}^{(y_1+y_2-s)^+}\theta\left(\frac{u}{s}\right)du-\int_{(x_1-s)^+}^{(x_1+x_2-s)^+}\theta\left(\frac{u}{s}\right)du\right|}{|y-x|}\\&=\sup_{x,y\in\mathbb{R}^2}\frac{\left|\int_{(x_1+x_2-s)^+}^{(y_1+y_2-s)^+}\theta\left(\frac{u}{s}\right)du-\int_{(x_1-s)^+}^{(y_1-s)^+}\theta\left(\frac{u}{s}\right)du\right|}{|y-x|}\\&\le\sup_{x,y\in\mathbb{R}^2}\frac{M|y_1+y_2-x_1-x_2|}{|y-x|}+\frac{M|y_1-x_1|}{|y-x|}\le(1+\sqrt{2})M<\infty.\end{aligned}$$

Next, we show that (23) hold, i.e., $\|\alpha_t^n(x,i)\|$ is Lipschitz, for $i=5,6$. Note that for $a\ge 0$, $na-s^n-1\le\lfloor na-s^n\rfloor\le na-s^n$, which implies that $\frac{1}{n}\lfloor na-s^n\rfloor\to a-s$ as $n\to\infty$. Thus, for $x,y\in\mathbb{R}$,
$$\frac{\lfloor ny-s^n\rfloor-\lfloor nx-s^n\rfloor}{n(y-x)}\to 1\text{ as }n\to\infty.$$

Then there exists a constant n_1 such that for $n \geq n_1$,

$$\frac{\lfloor ny - s^n \rfloor - \lfloor nx - s^n \rfloor}{n(y-x)} \leq 2.$$

Similarly, we can find a constant n_2 such that for $n \geq n_2$,

$$\frac{\lfloor ny - nx \rfloor}{n(y-x)} \leq 2.$$

For $y > x \geq 0$, $n \geq n^* := \max\{n_1, n_2\}$,

$$\left(\lfloor ny - s^n \rfloor^+ - \lfloor nx - s^n \rfloor^+\right)^+ = \begin{cases} \lfloor ny - s^n \rfloor - \lfloor nx - s^n \rfloor \leq 2n(y-x) & \text{if } y > x > \frac{s^n}{n}, \\ \lfloor ny - s^n \rfloor \leq \lfloor ny - nx \rfloor \leq 2n(y-x) & \text{if } y > \frac{s^n}{n} \geq x, \\ 0 & \text{otherwise.} \end{cases}$$

That is, for $x, y \in \mathbb{R}, n \geq n^*$, we have

$$\left(\lfloor ny - s^n \rfloor^+ - \lfloor nx - s^n \rfloor^+\right)^+ \leq 2n|y-x|.$$

Therefore,

$$\begin{aligned} \|\alpha_t^n(x,5)\| &= \sup_{x,y \in \mathbb{R}^2} \frac{\left|\sum_{i=1}^{\lfloor ny_1 - s^n \rfloor^+} \theta\left(\frac{i}{s^n}\right) - \sum_{i=1}^{\lfloor nx_1 - s^n \rfloor^+} \theta\left(\frac{i}{s^n}\right)\right|}{|y-x|} \\ &\leq \sup_{x,y \in \mathbb{R}^2} \frac{M\left(\lfloor ny_1 - s^n \rfloor^+ - \lfloor nx_1 - s^n \rfloor^+\right)^+}{|y-x|} \leq \sup_{x,y \in \mathbb{R}^2} \frac{nM|y_1 - x_1|}{|y-x|} \leq nM. \end{aligned}$$

$$\begin{aligned} \|\alpha_t^n(x,6)\| &= \sup_{x,y \in \mathbb{R}^2} \frac{\left|\sum_{i=\lfloor ny_1 - s^n \rfloor^+ + 1}^{\lfloor ny_1 + ny_2 - s^n \rfloor^+} \theta\left(\frac{i}{s^n}\right) - \sum_{i=\lfloor nx_1 - s^n \rfloor^+ + 1}^{\lfloor nx_1 + nx_2 - s^n \rfloor^+} \theta\left(\frac{i}{s^n}\right)\right|}{|y-x|} \\ &= \sup_{x,y \in \mathbb{R}^2} \frac{\left|\sum_{i=\lfloor nx_1 + nx_2 - s^n \rfloor^+ + 1}^{\lfloor ny_1 + ny_2 - s^n \rfloor^+} \theta\left(\frac{i}{s^n}\right) - \sum_{i=\lfloor nx_1 - s^n \rfloor^+ + 1}^{\lfloor ny_1 - s^n \rfloor^+} \theta\left(\frac{i}{s^n}\right)\right|}{|y-x|} \\ &\leq \sup_{x,y \in \mathbb{R}^2} \frac{M\left(\lfloor ny_1 + ny_2 - s^n \rfloor^+ - \lfloor nx_1 + nx_2 - s^n \rfloor^+\right)^+}{|y-x|} + \frac{M\left(\lfloor ny_1 - s^n \rfloor^+ - \lfloor nx_1 - s^n \rfloor^+\right)^+}{|y-x|} \\ &\leq \sup_{x,y \in \mathbb{R}^2} \frac{nM|y_1 + y_2 - x_1 - x_2|}{|y-x|} + \frac{nM|y_1 - x_1|}{|y-x|} \leq n(1+\sqrt{2})M. \end{aligned}$$

Lastly, we show (24), i.e., the convergence of $\frac{\alpha_t^n(x,i)}{n}$ to $\alpha_t(x,i)$, for $i = 5, 6$. To do so, we first prove the following useful equation:

$$\frac{1}{n} \sum_{i=1}^{\lfloor nx - s^n \rfloor^+} \theta\left(\frac{i}{s^n}\right) \to \int_0^{(x-s)^+} \theta\left(\frac{u}{s}\right) du, \forall x \in \mathbb{R}. \tag{25}$$

Indeed, since $\frac{s^n}{n} \leq s$, we have $\lfloor nx - s^n \rfloor^+ \geq \lfloor s^n\left(\frac{x-s}{s}\right) \rfloor^+$ and

$$\frac{1}{n} \sum_{i=1}^{\lfloor nx - s^n \rfloor^+} \theta\left(\frac{i}{s^n}\right) = \frac{s^n}{n} \sum_{x \in \mathcal{P}_n} \frac{1}{s^n} \theta(x) + \frac{1}{n} \sum_{x \in \mathcal{Q}_n} \theta(x),$$

where

$$\begin{aligned} \mathcal{P}_n &= \{\frac{i}{s^n} | i \in \mathbb{Z}, 0 \leq i < s^n \left(\frac{x-s}{s}\right)^+\} = \{\frac{i}{s^n} | i \in \mathbb{Z}, 0 \leq i \leq \left\lfloor s^n \left(\frac{x-s}{s}\right) \right\rfloor^+\}, \\ \mathcal{Q}_n &= \{\frac{i}{s^n} | i \in \mathbb{Z}, s^n \left(\frac{x-s}{s}\right)^+ \leq i \leq \lfloor nx - s^n \rfloor^+\} = \{\frac{i}{s^n} | i \in \mathbb{Z}, \left\lceil s^n \left(\frac{x-s}{s}\right) \right\rceil^+ \leq i \leq \lfloor nx - s^n \rfloor^+\}. \end{aligned}$$

Note that,

$$\lim_{n\to\infty}\frac{|\mathcal{Q}_n|}{n}=\lim_{n\to\infty}\frac{\lfloor nx-s^n\rfloor^+-s^n\left(\frac{x-s}{s}\right)^+}{n}\le\lim_{n\to\infty}\frac{(nx-s^n)^+-s^n\left(\frac{x-s}{s}\right)^+}{n}$$
$$=\lim_{n\to\infty}\frac{s^n}{n}\left(\frac{x}{s^n/n}-1\right)^+-\lim_{n\to\infty}\frac{s^n}{n}\left(\frac{x-s}{s}\right)^+=0.$$

Therefore,

$$\frac{1}{n}\sum_{x\in\mathcal{Q}_n}\theta(x)\le\frac{M|\mathcal{Q}_n|}{n}\to 0.$$

Moreover, by the convergence of the Riemann sum, we can obtain that:

$$\frac{s^n}{n}\sum_{x\in\mathcal{P}_n}\frac{1}{s^n}\theta(x)+\theta\left(\left(\frac{x-s}{s}\right)^+\right)\left(\left(\frac{x-s}{s}\right)^+-\left(\frac{1}{s^n}\left\lfloor s^n\left(\frac{x-s}{s}\right)\right\rfloor^+\right)\right)\to s\int_0^{\frac{(x-s)^+}{s}}\theta(u)du. \tag{26}$$

Noting that the second term on the left of (26) converges to 0 as $n\to\infty$. Thus,

$$\frac{1}{n}\sum_{i=1}^{\lfloor nx-s^n\rfloor^+}\theta\left(\frac{i}{s^n}\right)\to s\int_0^{\frac{(x-s)^+}{s}}\theta(u)du=\int_0^{(x-s)^+}\theta\left(\frac{u}{s}\right)du.$$

By equation (25), we can obtain that $\frac{\alpha_t^n(x,5)}{n}\to\alpha_t(x,5)$, and $\frac{\alpha_t^n(x,5)}{n}+\frac{\alpha_t^n(x,6)}{n}\to\alpha_t(x,5)+\alpha_t(x,6)$, which further implies that $\frac{\alpha_t^n(x,6)}{n}\to\alpha_t(x,6)$. The proof is complete. □

Appendix B: Stability Theorems and Proofs for Section 5

In this appendix, we present the proofs of results in Section 5. Specifically, we introduce the concepts and theorems we use for the proofs of Proposition 1 and Theorem 2 in Appendices B.1 and B.3; and provide the proofs of these two propositions in Appendices B.2 and B.4, respectively.

B.1. Pre-requisites for the Proof of Proposition 1

In this section, we introduce concepts and theorems we use for the existence proof of periodic equilibrium in Proposition 1. We first obtain the definition and property of the Poincaré map.

Definition 4. Consider a single differential equation $\dot{x}=f(t,x)$ and assume that $f(t,x)$ is periodic in t with period T, for $x\in\mathbb{R}$. The **Poincaré map** associated with $\dot{x}=f(t,x)$ is the map $\phi(x_0)=x_1$, where $x(t)$ is the solution of the ODE with $x(0)=x_0, x_1=x(T)$.

The Poincaré map is monotone as shown in the following proposition.

Proposition 9. *Let $\phi:J\to\mathbb{R}$ be the Poincaré map for $\dot{x}=f(t,x)$, where J is an interval. Then, for $a,b\in J$, we have $a<b\Rightarrow\phi(a)<\phi(b)$.*

The next theorem provides conditions for the existence of fixed point of a function.

Theorem 4 **(One-dimensional Brouwer fixed-point theorem)**. *Every continuous function from a closed interval into itself has a fixed point.*

The following theorem gives a set of conditions under which an initial value problem has a unique solution.

Theorem 5 **(Picard's existence theorem)**. *Consider the initial value problem $\dot{x}=f(t,x)$, $x(t_0)=x_0$. Suppose $f(t,x)$ is uniformly Lipschitz continuous in x and continuous in t, then for some value $\epsilon>0$, there exists a unique solution x to the initial value problem on the interval $[t_0-\epsilon,t_0+\epsilon]$.*

B.2. Proof of Proposition 1

We use notation $\uparrow$ for increasing and convergence, i.e., $s^n/n \uparrow s$. We start from the no information case, i.e., $I = N$. We begin with a single-class version of the original theorem. That is, we show that if there exist a solution $x^N(t) \in \mathbb{R}$ to the following one-dimensional ordinary differential equation such that $\limsup_{t\to\infty} x^N(t) < \infty$, then there exist a periodic solution with the same period as $\lambda(t)$:

$$\dot{x}(t) = \lambda(t) - \mu(x(t) \wedge s) - \theta\left(\frac{\beta(x(t)-s)^+}{s}\right)(x(t)-s)^+ = f_N(t,x), \tag{27}$$

where $\lambda(t+d) = \lambda(t)$ for some $d > 0$. Note that, $f_N(t,x)$ is periodic in t with period d, i.e., for a given x, $f_N(t,x) = f_N(t+d,x)$.

First, we show that any solution to (27) starting from a finite initial condition is bounded. Note that, when $x(t) > s$, $f_N(t,x) = \lambda(t) - \mu s - \theta\left(\beta\frac{x(t)-s}{s}\right)(x(t)-s)$. Since $\lambda(t)$ is bounded and $\theta\left(\beta\frac{x-s}{s}\right)(x-s) \uparrow \infty$ as $x \to \infty$, then there exist a $\hat{x}$ such that $\theta\left(\beta\frac{\hat{x}-s}{s}\right)(\hat{x}-s) = \max_{t\ge 0}\lambda(t) - \mu s$, and $f_N(t,x) \le 0$ for $\forall x \ge \hat{x}$. Therefore, $x(t) \le \max\{\hat{x}, s\} < \infty$.

Next, let $x(t,\xi)$ be the solution to (27) with $x(0) = \xi \ge 0$. Then, (27) has a periodic solution with period d if, for every t, $x(t+d,\xi) = x(t,\xi)$. Note that $x(t+d,\xi) = x(t, x(d,\xi))$. Thus, it suffices to show that $\xi = x(d,\xi)$. We define the Poincaré map associated with the ODE as follows:

$$\phi(\xi) = x(d,\xi).$$

Then, showing that $\xi = x(d,\xi)$ amounts to showing that ξ is a fixed point of $\phi(\cdot)$. First, note that $\phi(\cdot)$ is a continuous function. We can obtain this continuous dependence on the initial conditions by applying Theorem 6.3.1 in Lebovitz (1999) since $f_N(t,x)$ is Lipschitz.

Thus, by Theorem 4, to show that $\phi(\cdot)$ has a fixed point, it suffices to show that there exists a finite closed interval $[\xi_1,\xi_2]$ such that $\phi([\xi_1,\xi_2]) \subset (\xi_1,\xi_2)$.

Define

$$\xi_1 := \inf\{x(kd), k \in \mathbb{N}\}$$

and

$$\xi_2 := \sup\{x(kd), k \in \mathbb{N}\}.$$

Then, by our previous analysis, we have that $\xi_1,\xi_2 < \infty$. If $\xi_1 = \xi_2$, then $x(t)$ is a constant, and so it is itself a periodic solution. If $\xi_1 < \xi_2$, then for $\xi \in (\xi_1,\xi_2)$, there exist k_1,k_2 such that

$$x(k_1 d) < \xi < x(k_2 d).$$

Thus, by the monotonicity of the Poincaré map, i.e., Proposition 9:

$$\phi(\xi) = x(d,\xi) < x(d, x(k_2 d)) = x(d + k_2 d, x(0)) = x((k_2+1)d) \le \xi_2.$$

We can also show that $\phi(\xi) \ge \xi_1$ as follows:

$$\phi(\xi) = x(d,\xi) > x(d, x(k_1 d)) = x(d + k_1 d, x(0)) = x((k_1+1)d) \ge \xi_1.$$

Thus, $\xi_1 \le \phi(\xi) \le \xi_2$. By the continuity of $\phi(\cdot)$, it follows that

$$\phi([\xi_1, \xi_2]) \subset (\xi_1, \xi_2),$$

which completes the proof. That is, there exist a periodic solution with period d to (27).

We note that $f_N(t,x)$ is uniformly (in t) Lipschitz continuous in x, so that the periodic solution to (27), which we refer to as $\tilde{x}(t)$, must be unique by Theorem 5, i.e., the Picard's existence theorem.

Now, we go back to the two-dimensional system of ordinary differential equations (7) and (8). For any solution $(x_1^N(t),\, x_2^N(t))$ of (7) and (8) such that $x_1^N(0)+x_2^N(0)=\tilde{x}_0(0)$, let $\lambda(t)=\lambda_1(t)+\lambda_2(t)$, then $x_0^N(t)=x_1^N(t)+x_2^N(t)$ is a solution to (27). Moreover, since the unique solution to (27) with initial condition $\tilde{x}_0(0)$ is $\tilde{x}_0(t)$, we must have $x_0^N(t)=\tilde{x}_0(t)$. That is, any solution $x(t)$ of (7) and (8) with initial conditions sum up to $\tilde{x}_0(0)$ must satisfy $x_1(t)+x_2(t)=\tilde{x}_0(t)$.

Now, consider (7) with $\tilde{x}_0(t)$ plugged in, i.e.,

$$\dot{x}_1(t)=\lambda_1(t)-\mu(x_1(t)\wedge s)-\theta\left(\frac{\beta(\tilde{x}_0(t)-s)^+}{s}\right)(x_1(t)-s)^+. \tag{28}$$

Recall that $\lambda(t)$ is periodic with period d, thus $\tilde{x}_0(t)$ is also periodic with period d, regard $\tilde{x}_0(t)$ as given and using a similar analysis to (28) as we find $\tilde{x}_0(t)$ in the first step, it must be that there exists a solution $\tilde{x}_1(t)$ to (28) which is periodic with period d. Let $\tilde{x}_2(t)$ be a solution to (7) and (8) with initial condition $\tilde{x}_2(0)=\tilde{x}_0(0)-\tilde{x}_1(0)$, then we must have $\tilde{x}_2(t)=\tilde{x}_0(t)-\tilde{x}_1(t)$. Since both $\tilde{x}_0(t)$ and $\tilde{x}_1(t)$ are periodic in d, $\tilde{x}_2(t)$ is also periodic in d, and the result follows.

For the full information case, by (7), (9), and the one-dimensional proof above, we can obtain the unique periodic solution with period d_1. Denote the solution as $\tilde{x}_1^I$, and plug into (8), for $I=F$. Then, the result follows from a similar argument as the last part of the proof for $I=N$. $\square$

B.3. Stability Theorem for Time-Varying System

THEOREM 6. ***Theorem 4.3 in Khalil (2014)*** *Let $y=(0,0)$ be an equilibrium point for (11)–(12), and $V:[0,\infty)\times\mathbb{R}^2\to\mathbb{R}$ be a continuous differentiable function such that*

$$W_1(y)\le V(t,y)\le W_2(y), \tag{29}$$

$$\frac{\partial V}{\partial t}+\frac{\partial V}{\partial y}g(t,y)\le -W_3(y),\ \text{for } \forall t\ge 0, \forall y\in\mathbb{R}^2, \tag{30}$$

where $W_i(y)$ are continuous positive definite functions on $\mathbb{R}^2$ for $i=1,2,3$. If $W_1(y)$ is radially unbounded, then $y=(0,0)$ is globally uniformly asymptotically stable.

Note that, a function $f(y):\mathbb{R}^2\to\mathbb{R}$ is **radially unbounded** if $|y|\to\infty\Rightarrow f(y)\to\infty$.

B.4. Proof of Theorem 2

We prove this theorem using Theorem 6. We show the proof for $I=N$, and the case for $I=F$ is similar to this case, so we omit it for brevity.

To apply Theorem 6, we need to find a continuous differentiable function $V(t,y)$ and continuous positive definite functions $W_i(y_1,y_2)$ for $i=1,2,3$, such that equations (29) and (30) are satisfied for $t\ge 0, \tilde{x}^N(t)\in\mathbb{R}_+^2, y\in\mathbb{R}^2$ such that $\tilde{x}^N+y\in\mathbb{R}_+^2$.

Let

$$V(t,y) = W_1(y) = W_2(y) = \frac{1}{2}y_1^2 + \frac{1}{2}(y_1+y_2)^2.$$

Then, (29) always holds, and it is clear that $W_1(y)$ is radially unbounded. Let

$$W_3(y) = \min\{\theta(0),\mu\}y_1^2 + \min\{\theta(0),\mu\}(y_1+y_2)^2.$$

The only thing left is to show that (30) holds, i.e.,

$$\dot{V}(y) = y_1\tilde{g}_1^N(y) + (y_1+y_2)(\tilde{g}_1^N(y) + \tilde{g}_2^N(y)) \le -W_3(y).$$

By system of equations (7) and (8) we can obtain that

$$\begin{aligned}
\tilde{g}_1^N(y) &= \mu_1(\tilde{x}_1^N(t)\wedge s) - \mu_1((\tilde{x}_1^N(t)+y_1)\wedge s) \\
&\quad + \theta\left(\frac{\beta(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s)^+}{s}\right)(\tilde{x}_1^N(t)-s)^+ \\
&\quad - \theta\left(\frac{\beta(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s)^+}{s}\right)(\tilde{x}_1^N(t)+y_1-s)^+, \\
\tilde{g}_2^N(y) &= \mu_2((s-\tilde{x}_1^N(t))^+\wedge\tilde{x}_2^N(t)) - \mu_2((s-\tilde{x}_1^N(t)-y_1)^+\wedge(\tilde{x}_2^N(t)+y_2)) \\
&\quad + \theta\left(\frac{\beta(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s)^+}{s}\right)(\tilde{x}_2^N(t)-(s-\tilde{x}_1^N(t))^+)^+ \\
&\quad - \theta\left(\frac{\beta(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s)^+}{s}\right)(\tilde{x}_2^N(t)+y_2-(s-\tilde{x}_1^N(t)-y_1)^+)^+.
\end{aligned}$$

For $(y_1,y_2)\neq(0,0)$, we show (30) by cases as follows:

1. When $\tilde{x}_1^N(t)+\tilde{x}_2^N(t)\le s, \tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2\le s, \tilde{x}_1^N(t)+y_1\le s$, then

$$\tilde{g}_1^N(y) = -\mu y_1, \quad \tilde{g}_2^N(y) = -\mu y_2.$$

Thus,

$$\dot{V}(y) = -\mu y_1^2 - \mu(y_1+y_2)^2 \le -W_3(y).$$

2. When $\tilde{x}_1^N(t)+\tilde{x}_2^N(t)\le s, \tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2> s, \tilde{x}_1^N(t)+y_1\le s$, then $y_1+y_2>0$, and

$$\begin{aligned}
\tilde{g}_1^N(y) &= -\mu y_1, \\
\tilde{g}_2^N(y) &= \mu(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1-s) \\
&\quad - \theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s}{s}\right)(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s).
\end{aligned}$$

Then,

$$\begin{aligned}
&(y_1+y_2)(\tilde{g}_1^N(y)+\tilde{g}_2^N(y)) \\
&= (y_1+y_2)(\mu(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s) - \theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s}{s}\right)(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s)) \\
&\le (y_1+y_2)(\mu(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s) - \theta(0)(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s)) \\
&= (y_1+y_2)(\mu-\theta(0))(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s) - \theta(0)(y_1+y_2)^2 \\
&\le \begin{cases} -(y_1+y_2)^2(\mu-\theta(0)) - \theta(0)(y_1+y_2)^2 = -\mu(y_1+y_2)^2 & \text{if } \theta(0)\ge\mu \\ -\theta(0)(y_1+y_2)^2 & \text{if } \theta(0)<\mu \end{cases} \\
&= -\min\{\theta(0),\mu\}(y_1+y_2)^2.
\end{aligned}$$

Thus,

$$\dot{V}(y) \le -\mu y_1^2 - \min\{\theta(0),\mu\}(y_1+y_2)^2 \le -W_3(y).$$

3. When $\tilde{x}_1^N(t)+\tilde{x}_2^N(t)\le s, \tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2>s, \tilde{x}_1^N(t)+y_1>s$, then $y_1>0$, and

$$\tilde{g}_1^N(y)=\mu(\tilde{x}_1^N(t)-s)-\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s}{s}\right)(\tilde{x}_1^N(t)+y_1-s),$$
$$\tilde{g}_2^N(y)=\mu\tilde{x}_2^N(t)-\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s}{s}\right)(\tilde{x}_2^N(t)+y_2).$$

Note that, by a similar analysis as the previous cases, we can obtain that

$$\begin{aligned}
y_1\tilde{g}_1^N(y)&\le(s-\tilde{x}_1^N(t))(\mu-\theta(0))y_1-\theta(0)y_1^2\\
&\le-\min\{\theta(0),\mu\}y_1^2.\\
(y_1+y_2)(\tilde{g}_1^N(y)+\tilde{g}_2^N(y))&\le(y_1+y_2)(\mu-\theta(0))(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s)-\theta(0)(y_1+y_2)^2\\
&\le-\min\{\theta(0),\mu\}(y_1+y_2)^2.
\end{aligned}$$

Thus,

$$\dot{V}(y)\le-\min\{\theta(0),\mu\}y_1^2-\min\{\theta(0),\mu\}(y_1+y_2)^2=-W_3(y).$$

4. When $\tilde{x}_1^N(t)+\tilde{x}_2^N(t)>s, \tilde{x}_1^N(t)\le s, \tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2\le s, \tilde{x}_1^N(t)+y_1\le s$, then $y_1+y_2<0$, and

$$\begin{aligned}
&\tilde{g}_1^N(y)=-\mu y_1,\\
&\tilde{g}_2^N(y)=-\mu(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_2-s)+\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s}{s}\right)(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s).
\end{aligned}$$

Then,

$$\begin{aligned}
(y_1+y_2)(\tilde{g}_1^N(y)+\tilde{g}_2^N(y))&\le-\mu(y_1+y_2)^2+(y_1+y_2)(\theta(0)-\mu)(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s)\\
&\le-\min\{\theta(0),\mu\}(y_1+y_2)^2.
\end{aligned}$$

Thus,

$$\dot{V}(y)\le-\mu y_1^2-\min\{\theta(0),\mu\}(y_1+y_2)^2\le-W_3(y).$$

5. When $\tilde{x}_1^N(t)+\tilde{x}_2^N(t)>s, \tilde{x}_1^N(t)\le s, \tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2>s, \tilde{x}_1^N(t)+y_1\le s$, then

$$\begin{aligned}
\tilde{g}_1^N(y)&=-\mu y_1,\\
\tilde{g}_2^N(y)&=\mu y_1+\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s}{s}\right)(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s)\\
&\quad-\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s}{s}\right)(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s).
\end{aligned}$$

Then,

$$\begin{aligned}
&(y_1+y_2)(\tilde{g}_1^N(y)+\tilde{g}_2^N(y))\\
&\le(y_1+y_2)\left(\left(\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s}{s}\right)-\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s}{s}\right)\right)(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s)\right)-\theta(0)(y_1+y_2)^2\\
&\le-\theta(0)(y_1+y_2)^2.
\end{aligned}$$

Thus,

$$\dot{V}(y)\le-\mu y_1^2-\theta(0)(y_1+y_2)^2\le-W_3(y).$$

6. When $\tilde{x}_1^N(t)+\tilde{x}_2^N(t)>s, \tilde{x}_1^N(t)\le s, \tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2>s, \tilde{x}_1^N(t)+y_1>s$, then $y_1>0$, and

$$\begin{aligned}\tilde{g}_1^N(y)&=\mu(\tilde{x}_1^N(t)-s)-\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s}{s}\right)(\tilde{x}_1^N(t)+y_1-s),\\ \tilde{g}_2^N(y)&=\mu(s-\tilde{x}_1^N(t))+\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s}{s}\right)(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s)\\ &\quad-\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s}{s}\right)(\tilde{x}_2^N(t)+y_2)^+.\end{aligned}$$

Then,

$$y_1\tilde{g}_1^N(y)\le(\mu-\theta(0))(\tilde{x}_1^N(t)-s)y_1-\theta(0)y_1^2\le-\min\{\mu,\theta(0)\}y_1^2.$$

$$\begin{aligned}(y_1+y_2)(\tilde{g}_1^N(y)+\tilde{g}_2^N(y))&=(y_1+y_2)\Bigg(\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s}{s}\right)(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s)\\ &\quad-\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s}{s}\right)(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s)\Bigg)\\ &\le-\theta(0)(y_1+y_2)^2.\end{aligned}$$

Thus,

$$\dot{V}(y)\le-\min\{\mu,\theta(0)\}y_1^2-\theta(0)(y_1+y_2)^2\le-W_3(y).$$

7. When $\tilde{x}_1^N(t)>s, \tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2\le s, \tilde{x}_1^N(t)+y_1\le s$, then $y_1<0, y_1+y_2<0$, and

$$\begin{aligned}\tilde{g}_1^N(y)&=\mu(s-\tilde{x}_1^N(t)-y_1)+\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s}{s}\right)(\tilde{x}_1^N(t)-s),\\ \tilde{g}_2^N(y)&=-\mu(\tilde{x}_2^N(t)+y_2)+\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s}{s}\right)\tilde{x}_2^N(t).\end{aligned}$$

Then,

$$\begin{aligned}y_1\tilde{g}_1^N(y)&\le(\theta(0)-\mu)(\tilde{x}_1^N(t)-s)y_1-\mu y_1^2\le-\min\{\mu,\theta(0)\}y_1^2.\\ (y_1+y_2)(\tilde{g}_1^N(y)+\tilde{g}_2^N(y))&=-\mu(y_1+y_2)^2+(y_1+y_2)\left(\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s}{s}\right)-\mu\right)(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s)\\ &\le-\min\{\mu,\theta(0)\}(y_1+y_2)^2.\end{aligned}$$

Thus,

$$\dot{V}(y)\le-\min\{\mu,\theta(0)\}y_1^2-\min\{\mu,\theta(0)\}(y_1+y_2)^2=-W_3(y).$$

8. When $\tilde{x}_1^N(t)>s, \tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2>s, \tilde{x}_1^N(t)+y_1\le s$, then $y_1<0$, and

$$\begin{aligned}\tilde{g}_1^N(y)&=\mu(s-\tilde{x}_1^N(t)-y_1)+\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s}{s}\right)(\tilde{x}_1^N(t)-s),\\ \tilde{g}_2^N(y)&=-\mu(s-\tilde{x}_1^N(t)-y_1)+\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s}{s}\right)\tilde{x}_2^N(t)\\ &\quad-\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s}{s}\right)(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s).\end{aligned}$$

Then,

$$y_1\tilde{g}_1^N(y)\le(\theta(0)-\mu)(\tilde{x}_1^N(t)-s)y_1-\mu y_1^2\le-\min\{\mu,\theta(0)\}y_1^2.$$

$$
\begin{aligned}
(y_1+y_2)(\tilde{g}_1^N(y)+\tilde{g}_2^N(y)) = (y_1+y_2)\Bigg(&\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s}{s}\right)(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s) \\
&-\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s}{s}\right)(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s)\Bigg) \\
\le{}& -\theta(0)(y_1+y_2)^2.
\end{aligned}
$$

Thus,

$$
\dot{V}(y) \le -\min\{\mu, \theta(0)\}y_1^2 - \theta(0)(y_1+y_2)^2 \le -W_3(y).
$$

9. When $\tilde{x}_1^N(t) > s, \tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2 > s, \tilde{x}_1^N(t)+y_1 > s$,

$$
\begin{aligned}
\tilde{g}_1^N(y) &= \theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s}{s}\right)(\tilde{x}_1^N(t)-s) \\
&\quad -\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s}{s}\right)(\tilde{x}_1^N(t)+y_1-s), \\
\tilde{g}_2^N(y) &= \theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s}{s}\right)\tilde{x}_2^N(t) - \theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s}{s}\right)(\tilde{x}_2^N(t)+y_2).
\end{aligned}
$$

(a) If $y_1(y_1+y_2) \ge 0$,

$$
\begin{aligned}
y_1\tilde{g}_1^N(y) &\le \left(\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s}{s}\right) - \theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s}{s}\right)\right)(\tilde{x}_1^N(t)-s)y_1 - \theta(0)y_1^2 \\
&\le -\theta(0)y_1^2.
\end{aligned}
$$

$$
\begin{aligned}
(y_1+y_2)(\tilde{g}_1^N(y)+\tilde{g}_2^N(y)) = (y_1+y_2)\Bigg(&\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s}{s}\right)(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s) \\
&-\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s}{s}\right)(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s)\Bigg) \\
\le{}& -\theta(0)(y_1+y_2)^2.
\end{aligned}
$$

Thus,

$$
\dot{V}(y) \le -\theta(0)y_1^2 - \theta(0)(y_1+y_2)^2 \le -W_3(y).
$$

(b) If $y_1(y_1+y_2) < 0$, then

$$
\begin{aligned}
\dot{V}(y) &= y_1\tilde{g}_1^N(y) + (y_1+y_2)(\tilde{g}_1^N(y)+\tilde{g}_2^N(y)) \\
&= \theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s}{s}\right)(y_1(\tilde{x}_1^N(t)-s)+(y_1+y_2)(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s)) \\
&\quad -\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s}{s}\right)(y_1(\tilde{x}_1^N(t)+y_1-s)+(y_1+y_2)(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s)) \\
&= \left(\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s}{s}\right) - \theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s}{s}\right)\right) \\
&\quad \times (y_1(\tilde{x}_1^N(t)-s)+(y_1+y_2)(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s)) \\
&\quad -\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s}{s}\right)(y_1^2+(y_1+y_2)^2).
\end{aligned}
$$

i. If $2y_1+y_2 \le 0$, then

$$
\begin{aligned}
\dot{V}(y) &\le \left(\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s}{s}\right) - \theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s}{s}\right)\right)(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s)(2y_1+y_2) \\
&\quad -\theta(0)(y_1^2+(y_1+y_2)^2) \\
&\le \theta(0)(y_1^2+(y_1+y_2)^2) \le -W_3(y).
\end{aligned}
$$

ii. If $2y_1+y_2>0$, then

$$\begin{aligned}
\dot{V}(y) \le &\left(\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s}{s}\right)-\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s}{s}\right)\right)\\
&\times(y_1(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s)+(y_1+y_2)(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s))\\
&-\left(\theta\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s}{s}\right)-\theta(0)\right)(y_1^2+(y_1+y_2)^2)-\theta(0)(y_1^2+(y_1+y_2)^2)\\
&\le\theta'\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s}{s}\right)(-\beta\frac{y_1+y_2}{s})\\
&\times(y_1(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s)+(y_1+y_2)(\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s))\\
&-\theta'\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s}{s}\right)\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s}{s}\right)(y_1^2+(y_1+y_2)^2)-W_3(y)\\
&=\theta'\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s}{s}\right)\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s}{s}\right)(-y_1(y_1+y_2)-y_1^2-(y_1+y_2)^2)\\
&-\theta'\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s}{s}\right)\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)-s}{s}\right)(y_1+y_2)^2-W_3(y)\\
&\le-\theta'\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s}{s}\right)\left(\beta\frac{\tilde{x}_1^N(t)+\tilde{x}_2^N(t)+y_1+y_2-s}{s}\right)\left(3y_1^2+3y_1y_2+y_2^2\right)-W_3(y)\\
&\le-W_3(y).
\end{aligned}$$

Note that we used above the fact that:

A. When $y_1(y_1+y_2)<0$, we have $\tilde{x}_2^N(t)+y_1+y_2\ge 0$. In particular, if $y_1>0, y_1+y_2<0$, then $\tilde{x}_2^N(t)+y_1+y_2\ge\tilde{x}_2^N(t)+y_2\ge 0$; if $y_1<0, y_1+y_2>0$, then $\tilde{x}_2^N(t)+y_1+y_2\ge\tilde{x}_2^N(t)\ge 0$.

B. For concave and and differentiable function θ and $a,b\in\mathbb{R}$, we have $\theta(b)\le\theta(a)+\theta'(a)(b-a)$.

C. $3y_1^2+3y_1y_2+y_2^2=3(y_1+\frac{1}{2}y_2)^2+\frac{1}{4}y_2^2\ge 0$.

□

Appendix C: Proofs and Supplementary Results for Section 6

In this appendix, we provide the proofs and supplementary results for Section 6. In particular, in Appendices C.1 and C.2, we present concepts and theorems that facilitate the proofs in this section; in Appendices C.3–C.5, we provide the proofs for Sections 6.4–6.6, respectively.

C.1. Comparison Theorems of Ordinary Differential Equations

The following lemmas are useful for the comparison of the trajectories of the fluid model $x^I(t)$ under different information levels.

LEMMA 1. ***Proposition 6.4 in Bagagiolo (2012)*** *Let $f,g:A\to\mathbb{R}, A\subseteq\mathbb{R}^2$ open, be Lipschitz continuous in x, such that $f(t,x)\le g(t,x),\forall(t,x)\in A$. Then, if $y,z:I\to\mathbb{R}$ are, respectively, the solutions of the following two initial value problems:*

$$y'(t)=f(t,y), y(t_0)=y_0,$$
$$z'(t)=g(t,z), z(t_0)=z_0,$$

where $y_0\le z_0$ and I is the common interval of existence, then $y(t)\le z(t)$ for $t\in I, t\ge t_0$.

Lemma 2. ***Strong Comparison Theorem in McNabb (1986)*** *Suppose $y(t)$, $z(t)$ are continuous on $[a,b]$ and differentiable on $(a,b]$, f is a continuous mapping from $\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ that satisfies a Lipschitz condition, and*

$$y(a) > z(a), \frac{dy}{dt} - f(t,y) \geq \frac{dz}{dt} - f(t,z) \text{ on } (a,b].$$

Then $y > z$ on $[a,b]$.

In the proofs we will rely on the following application of Lemma 2: Letting $f(t,y) = dy/dt, g(t,z) = dz/dt$, $a = t_0$, and $b = t_0 + d$, then Lemma 2 implies that:

$$y(t_0) > z(t_0) \text{ and } f(t,z) \geq g(t,z) \Rightarrow y(t) > z(t) \quad \forall t \in [t_0, t_0 + d]. \tag{31}$$

C.2. Continuity of $x^I(t)$ on parameters

Recall that $f_k^I(t, \tilde{x}^I(t))$ is the net flow rate of class k customers under information level I at time t in equilibrium, which depends on the arrival rates $\lambda_k(t)$. We obtain the continuity of $\tilde{x}^I(t)$ with respect to $\lambda_k(t)$ using the following theorem, and the continuity result is given in Corollary 1.

Before we proceed, we introduce the class of p**-norms** $||\cdot||_p$ for a vector $x \in \mathbb{R}$ in below:

$$||x||_p := (|x_1|^p + \ldots + |x_n|^p)^{1/p}, \quad 1 \leq p < \infty,$$

and

$$||x||_\infty := \max_i |x_i|.$$

Theorem 7. *(Theorem 3.4 in Khalil (2002)) Let $f(t,x)$ be piecewise continuous in t and Lipschitz in x on $[t_0, t_1] \times W$ with a Lipschitz constant L, where $W \subset R^n$ is an open connected set. Let $y(t)$ and $z(t)$ be solutions of*

$$\dot{y} = f(t,y), \quad y(t_0) = y_0$$

and

$$\dot{z} = f(t,z) + g(t,z), \quad z(t_0) = z_0$$

such that $y(t), z(t) \in W$ for all $t \in [t_0, t_1]$. Suppose that

$$||g(t,x)||_p \leq \sigma, \quad \forall (t,x) \in [t_0, t_1] \times W$$

for some $\sigma > 0$ and any p-norm. Then,

$$||y(t) - z(t)||_p \leq ||y_0 - z_0||_p e^{L(t-t_0)} + \frac{\sigma}{L}\left(e^{L(t-t_0)} - 1\right), \quad \forall t \in [t_0, t_1].$$

Corollary 1. *For $t \geq 0$, $\tilde{x}^I(t)$ is continuous in $(\lambda_1(t), \lambda_2(t))$, for $I \in \{F, N\}$.*

Proof of Corollary 1. For simplicity, let $\lambda(t) := (\lambda_1(t), \lambda_2(t))$. Consider two systems under information level I starting from the same initial $\tilde{x}_0^I$ at $t = 0$ with arrival rates $\lambda(t)$ and $\lambda'(t)$. Let $\tilde{x}^I(t)$ and $\tilde{x}'^I(t)$ be the corresponding number-in-system trajectories. We rewrite the net flow rate f_k^I as a function of $t, \tilde{x}^I(t)$, and $\lambda(t)$, for $k = 1, 2$, and denote $f^I := (f_1^I, f_2^I)$. Then $\tilde{x}^I(t)$ and $\tilde{x}'^I(t)$ are the solutions of the following two initial value problems, respectively:

$$\dot{y} = f^I(t, y, \lambda(t)), \quad y(0) = \tilde{x}_0^I.$$

$$\dot{z} = f^I(t, z, \lambda'(t)), \quad z(0) = \tilde{x}_0^I.$$

In particular, $f^I(t,x,\lambda'(t)) - f^I(t,x,\lambda(t)) = \lambda'(t) - \lambda(t)$, for $\forall x \geq \mathbf{0}$. Recall that, f^I is Lipschitz in $\tilde{x}^I(t)$ with a Lipschitz constant $L \leq 2\sqrt{2}M$. Applying Theorem 7 with $f(t,y) = f^I(t,y,\lambda(t)), g(t,z) = \lambda'(t) - \lambda(t)$, we have:

$$\text{If } ||\lambda'(t) - \lambda(t)||_p \leq \sigma, \text{ then } ||\tilde{x}'^I(t) - \tilde{x}^I(t)||_p \leq \frac{\sigma}{L}\left(e^{Lt} - 1\right).$$

Thus, for any $\epsilon > 0$ and fixed t, L, there exists $\sigma := \frac{\epsilon L}{e^{Lt}-1}$ such that, $||\tilde{x}'^I(t) - \tilde{x}^I(t)||_p \leq \epsilon$ when $||\lambda'(t) - \lambda(t)||_p \leq \sigma$. Since L is finite, and we have shown the existence and stability of periodic equilibrium for $\tilde{x}^I(t)$ and $\tilde{x}'^I(t)$, the result follows. $\square$

Lastly, we establish the continuity of $\tilde{x}^N(t,\boldsymbol{\beta})$ in $\boldsymbol{\beta}$ by rewriting Theorem 3.5 in Khalil (2002) as follows.

Proposition 10. *If $f(t,x,\boldsymbol{\beta})$ is continuous in $(t,x,\boldsymbol{\beta})$ and locally Lipschitz in x on $[t_0,t_1] \times R^n \times \{||\boldsymbol{\beta} - \boldsymbol{\beta}_0||_p \leq c\}$. Let $y(t,\boldsymbol{\beta}_0)$ be a solution of $\dot{x} = f(t,x,\boldsymbol{\beta}_0)$ with $y(t_0,\boldsymbol{\beta}_0) = y_0$. Then, given $\epsilon > 0$, there is $\delta > 0$ such that if*

$$||z_0 - y_0||_p < \delta \text{ and } ||\boldsymbol{\beta} - \boldsymbol{\beta}_0||_p < \delta,$$

then there is a unique solution $z(t,\boldsymbol{\beta})$ of $\dot{x} = f(t,x,\boldsymbol{\beta})$ with $z(t_0,\boldsymbol{\beta}) = z_0$, and $z(t,\boldsymbol{\beta})$ satisfies

$$||z(t,\boldsymbol{\beta}) - y(t,\boldsymbol{\beta}_0)||_p < \epsilon, \quad \forall t \in [t_0,t_1].$$

C.3. Proofs for Section 6.4

For single-class non-stationary systems, (13), (15), and (16) reduce to:

$$\bar{A}^I = \frac{1}{d}\int_0^d \lambda(t)dt - \frac{\mu}{d}\int_0^d (\tilde{x}^I(t) \wedge s)dt, \text{ for } I \in \{F,N\}, \tag{32}$$

$$\bar{A}^N(\beta) = \frac{1}{d}\int_0^d \theta\left(\beta\frac{(\tilde{x}^N(t) - s)^+}{s}\right)(\tilde{x}^N(t) - s)^+ dt, \tag{33}$$

$$\bar{A}^F = \frac{1}{d}\int_0^d \int_0^{(\tilde{x}^F(t)-s)^+} \theta\left(u/s\right) du dt. \tag{34}$$

To facilitate the proof, we introduce the following useful lemma without a proof.

Lemma 3. *Let a_1, a_2, b_1, b_2 be any real numbers, then*

$$\min\{a_1,a_2\} - \min\{b_1,b_2\} \leq \max\{a_1 - b_1, a_2 - b_2\}.$$

Then, by (32) we have

$$\bar{A}^N(\beta) - \bar{A}^F \leq \frac{1}{d}\int_0^d \left(\tilde{x}^F(t) - \tilde{x}^N(t,\beta)\right)^+ dt, \tag{35}$$

$$\bar{A}^F - \bar{A}^N(\beta) \leq \frac{1}{d}\int_0^d \left(\tilde{x}^N(t,\beta) - \tilde{x}^F(t)\right)^+ dt. \tag{36}$$

We prove Propositions 3 (average number-in-system) and 4 (average system abandonment rates) by applying Lemma 3 and the following Lemma 4. Lemma 4 compares the equilibrium number-in-system processes.

Lemma 4. *For single-class systems with non-stationary periodic arrivals, the equilibrium number-in-system processes under no (N) and full (F) information compare as follows for all $t \geq 0$:*

1. *If* $\bar{\rho} \le 1$, *then* $\tilde{x}^N(t,\beta) = \tilde{x}^F(t) \le s$.
2. *If* $\underline{\rho} \ge 1$, *then* $s \le \tilde{x}^N(t,\beta), s \le \tilde{x}^F(t)$.
3. *If* $1 < \bar{\rho}$, *then* $\tilde{x}^N(t,\beta)$ *is continuously non-increasing in* β, *with strict decreasing if* $\max_{t\ge 0} \tilde{x}^N(t,0) > s$ *and:*
   (a) *If* $\beta \ge 0.5$, *then* $\tilde{x}^N(t,\beta) \le \tilde{x}^F(t)$ $\forall t$;
      i. *If* $\max_{t\ge 0} \tilde{x}^N(t,\beta) > s$, *then the inequality is strict* $\forall t$,
      ii. *If* $\max_{t\ge 0} \tilde{x}^F(t) > s$, *then* $\max_{t\ge 0} \tilde{x}^N(t,\beta) > s$.
   (b) *If* $\beta = 0$, *then* $\tilde{x}^N(t,0) \ge \tilde{x}^F(t)$ $\forall t$;
      i. *If* $\max_{t\ge 0} \tilde{x}^F(t) > s$, *then the inequality is strict* $\forall t$,
      ii. *If* $\max_{t\ge 0} \tilde{x}^N(t,0) > s$, *then* $\max_{t\ge 0} \tilde{x}^F(t) > s$.

Proof of Lemma 4. To facilitate the proof, we first recall the net flow rate $f^I(t,x)$ for $I = N, F$ as follows:

$$f^N(t,x,\beta) = \lambda(t) - \mu(x(t)\wedge s) - \theta\left(\frac{\beta(x(t)-s)^+}{s}\right)(x(t)-s)^+,$$
$$f^F(t,x) = \lambda(t) - \mu(x(t)\wedge s) - \int_0^{(x(t)-s)^+} \theta\,(u/s)\,du.$$

Note that for $t,x \ge 0$, when $\beta \ge 0.5$, $f^N(t,x,\beta) \le f^F(t,x)$; when $\beta = 0$, $f^N(t,x,\beta) \ge f^F(t,x)$. Also, by the balance equations (32)-(34) we have

$$0 = \int_0^d f^N(t,\tilde{x}^N,\beta)dt = \int_0^d f^F(t,\tilde{x}^F)dt. \tag{37}$$

1. If $\bar{\rho} \le 1$, since the arrival rate $\lambda(t)$ is non-stationary, then $\lambda(t) \le s\mu$, for $t \ge 0$, and $\lambda(t) < s\mu$ for some time interval(s) within each period. Therefore, we have $\frac{1}{d}\int_0^d \lambda(t)dt < s\mu$.

First, we show that $\tilde{x}^N(t,\beta) \le s$. Note that, there must exist $t_1 \in [0,d)$ such that $\tilde{x}^N(t_1,\beta) < s$. If otherwise, i.e., $\tilde{x}^N(t,\beta) \ge s$ for $t \in [0,d)$, then (33) implies that

$$\bar{A}^N(\beta) = \frac{1}{d}\int_0^d \theta\left(\frac{\beta(\tilde{x}^N(t,\beta)-s)^+}{s}\right)(\tilde{x}^N(t,\beta)-s)^+dt \ge 0.$$

However, by (32) we have

$$\bar{A}^N(\beta) = \frac{1}{d}\int_0^d \lambda(t)dt - s\mu < 0,$$

which yields a contradiction. Therefore, we can always find such t_1 with $\tilde{x}^N(t_1,\beta) < s$. Moreover, if $\tilde{x}^N(t,\beta) < s$ and $\tilde{x}^N(t,\beta) \to s$, by (7) we have

$$\dot{\tilde{x}}(t) = \lambda(t) - \mu\tilde{x}(t) \to \lambda(t) - \mu s \le 0.$$

That is, starting from t_1, $\tilde{x}^N(t,\beta)$ would never exceed s for $t \ge t_1$. Since $\tilde{x}^N(t,\beta)$ is a periodic equilibrium, we must have $\tilde{x}^N(t,\beta) \le s$ for $t \ge 0$.

By a similar analysis, we can show that $\tilde{x}^F(t) \le s$.

Once we obtain $\tilde{x}^N(t,\beta) \le s, \tilde{x}^F(t) \le s$, then by (33), we have $\bar{A}^I = 0$, for $I \in \{F,N\}$. By (7), $\tilde{x}^N(t,\beta)$ and $\tilde{x}^F(t)$ both uniquely solves the ODE

$$\dot{\tilde{x}}(t) = \lambda(t) - \mu\tilde{x}(t).$$

Hence, $\tilde{x}^N(t,\beta) = \tilde{x}^F(t) \le s$.

2. If $\underline{\rho} \geq 1$, then $\lambda(t) \geq s\mu$, for $t \geq 0$, and $\lambda(t) > s\mu$ for some time interval(s) within each period. Therefore, we have $\frac{1}{d}\int_0^d \lambda(t)dt > s\mu$. By a similar analysis as we show $\tilde{x}^N(t,\beta) \leq s$ for $\bar{\rho} \leq 1$, we can obtain that $\tilde{x}^N(t,\beta) \geq s$, $\tilde{x}^F(t) \geq s$, and the inequality is strict for some time interval(s) within each period (i.e., $\max \tilde{x}^I(t) > s$), using (32)-(34).

3. If $1 < \bar{\rho}$, first we show that $\tilde{x}^N(t,\beta)$ is continuously non-increasing in β. The continuity part is given by Proposition 10. We prove the monotonicity part by Lemma 1. Consider two systems under no information model, one with β^h and the other with β^l, where $\beta^h > \beta^l$. Then, $f^N(t,x,\beta^h) \leq f^N(t,x,\beta^l)$, for $t, x \geq 0$. Note that there must exist $t_2 \in [0,d)$ such that $\tilde{x}^N(t_2,\beta^h) \leq \tilde{x}^N(t_2,\beta^l)$. If otherwise, i.e., $\tilde{x}^N(t,\beta^h) > \tilde{x}^N(t,\beta^l)$ for $t \in [0,d)$, then $f^N(t,\tilde{x}^N(t,\beta^h),\beta^h) < f^N(t,\tilde{x}^N(t,\beta^l),\beta^l)$, for $t \in [0,d)$, which contradicts to (37). Let $y(t) = \tilde{x}^N(t,\beta^h)$, $z(t) = \tilde{x}^N(t,\beta^l)$, $f(t,y) = f^N(t,y,\beta^h)$, and $g(t,z) = f^N(t,z,\beta^l)$, then we have $\tilde{x}^N(t,\beta^h) \leq \tilde{x}^N(t,\beta^l)$ for $t \in [0,d)$ by Lemma 1. Due to the periodicity of $\tilde{x}^N$, we have $\tilde{x}^N(t,\beta^h) \leq \tilde{x}^N(t,\beta^l)$ for $t \geq 0$, i.e., $\tilde{x}^N(t,\beta)$ is non-increasing in β.

We show the strict monotonicity by Lemmas 3 and 2. To do so, we first show that $\tilde{x}^N(t,\beta^h) < \tilde{x}^N(t,\beta^l)$ for $t \geq 0$ when $\max_{t\geq 0} \tilde{x}^N(t,\beta^l) > s$. We need to show the existence of $t_2' \in [0,d)$ such that $\tilde{x}^N(t_2',\beta^h) < \tilde{x}^N(t_2',\beta^l)$. If otherwise, i.e., $\tilde{x}^N(t,\beta^h) \geq \tilde{x}^N(t,\beta^l)$ for $t \in [0,d)$, then since $\max_{t\geq 0} \tilde{x}^N(t,\beta^l) > s$, there must exists an interval $\eta_2 \subset [0,d)$ such that $\tilde{x}^N(t,\beta^h) \geq \tilde{x}^N(t,\beta^l) > s$ for $t \in \eta_2$. Then, $f^N(t,\tilde{x}^N(t,\beta^h),\beta^h) \leq f^N(t,\tilde{x}^N(t,\beta^l),\beta^l)$, for $t \in [0,d)$, and the inequality is strict for $t \in \eta_2$. This contradicts to $0 = \int_0^d f^N(t,\tilde{x}^N(t,\beta^h),\beta^h)dt = \int_0^d f^N(t,\tilde{x}^N(t,\beta^l),\beta^l)dt$ from (37). Thus, such t_2' exists and $\tilde{x}^N(t,\beta^h) < \tilde{x}^N(t,\beta^l)$ for all t when $\max_{t\geq 0} \tilde{x}^N(t,\beta^l) > s$ by Lemma 2.

Now we show that $\max_{t\geq 0} \tilde{x}^N(t,0) > s$ implies $\max_{t\geq 0} \tilde{x}^N(t,\beta^l) > s$, for any $\beta^l > 0$. Indeed, if $\max_{t\geq 0} \tilde{x}^N(t,0) > s$ and $\max_{t\geq 0} \tilde{x}^N(t,\beta^l) \leq s$, then by (33) we have $\bar{A}^N(0) > 0$ and $\bar{A}^N(\beta^l) = 0$. However, by Lemma 3 and (32) we have $\bar{A}^N(0) - \bar{A}^N(\beta^l) \leq \frac{1}{d}\int_0^d \left(\tilde{x}^N(t,\beta^l) - \tilde{x}^N(t,0)\right)^+ dt \leq 0$, which leads to a contradiction.

(a) If $\beta \geq 0.5$, recall that $f^N(t,x,\beta) \leq f^F(t,x)$, we show $\tilde{x}^N(t,\beta) \leq \tilde{x}^F(t)$ using Lemma 1. Let $y(t) = \tilde{x}^N(t,\beta)$, $z(t) = \tilde{x}^F(t)$, $f(t,y) = f^N(t,y,\beta)$, and $g(t,z) = f^F(t,z)$. Recall that $f^N(t,x,\beta) \leq f^F(t,x)$, to apply Lemma 1, we just need to identify a starting point $t_3 \in [0,d)$ such that $\tilde{x}^N(t_3,\beta) \leq \tilde{x}^F(t_3)$. If on the contrary, $\tilde{x}^N(t,\beta) > \tilde{x}^F(t)$ for $t \in [0,d)$, then

$$\begin{aligned} f^N(t,\tilde{x}^N,\beta) - f^F(t,\tilde{x}^F) = \mu(\tilde{x}^F(t) \wedge s) + \int_0^{(\tilde{x}^F(t)-s)^+} \theta\,(t/s)\,dt \\ - \mu(\tilde{x}^N(t,\beta) \wedge s) - \theta\left(\frac{(\tilde{x}^N(t,\beta)-s)^+}{s}\right)(\tilde{x}^N(t,\beta)-s)^+ < 0. \end{aligned} \tag{38}$$

However, the above inequality contradicts to the fact that $0 = \int_0^d f^N(t,\tilde{x}^N,\beta)dt = \int_0^d f^F(t,\tilde{x}^F)dt$. Therefore, there must exist such t_3 with $\tilde{x}^N(t_3,\beta) \leq \tilde{x}^F(t_3)$. Apply Lemma 1 we have $\tilde{x}^N(t,\beta) \leq \tilde{x}^F(t)$ for $t \geq t_2$. Since both $\tilde{x}^N(t,\beta)$ and $\tilde{x}^F(t)$ are periodic with period d, we have $\tilde{x}^N(t,\beta) \leq \tilde{x}^F(t)$ for $t \geq 0$.

i. Note that, (38) is strict when $\tilde{x}^N(t,\beta) \geq \tilde{x}^F(t)$ if and only if $\tilde{x}^N(t,\beta) > s$. Therefore, we can find the t_3 such that $\tilde{x}^N(t_3,\beta) < \tilde{x}^F(t_3)$ by contradiction if and only if $\max_{t\geq 0} \tilde{x}^N(t,\beta) > s$, and obtain $\tilde{x}^N(t,\beta) < \tilde{x}^F(t)$ for $t \geq 0$ by applying Lemma 2 with $y(t) = \tilde{x}^F(t)$, $z(t) = \tilde{x}^N(t,\beta)$, $f(t,z) = f^F(t,\tilde{x}^F)$, and $g(t,z) = f^N(t,\tilde{x}^N,\beta)$.

ii. Now we show that $\max_{t\geq 0}\tilde{x}^F(t)>s$ implies $\max_{t\geq 0}\tilde{x}^N(t,\beta)>s$. If not, i.e., if $\max_{t\geq 0}\tilde{x}^F(t)>s$ and $\max_{t\geq 0}\tilde{x}^N(t,\beta)\leq s$, then by (33) and (34) we have $\bar{A}^N(\beta)=0$ and $\bar{A}^F>0$. However, by (36), we have $\bar{A}^F-\bar{A}^N(\beta)\leq \frac{1}{d}\int_0^d(\tilde{x}^N(t,\beta)-\tilde{x}^F(t))^+\,dt\leq 0$, which yields a contradiction.

(b) If $\beta=0$, $f^N(t,x,\beta)\geq f^F(t,x)$, for $t,x\geq 0$. By a similar analysis as we show part (a), we can obtain the desired results. □

C.3.1. Proof of Proposition 3. Case 1 directly follows from Lemma 4.1. For Case 2, $\max_{t\geq 0}\tilde{x}^N(t,0)>s$ implies that $\max_{t\geq 0}\tilde{x}^N(t,\beta)>s$ by Lemma 4.3. Moreover,

1. For $\beta=0$ we have $\bar{x}^N(0)>\bar{x}^F$: Since $\max_{t\geq 0}\tilde{x}^N(t,0)>s$, this follows from Lemma 4.3.(b).
2. For $\beta\geq 0.5$, since $\max_{t\geq 0}\tilde{x}^N(t,\beta)>s$, Lemma 4.3.(a) implies that $\bar{x}^N(\beta)<\bar{x}^F$.
3. Lemma 4.3 implies that $\bar{x}^N(\beta)$ continuously and strictly decreases in β as long as $\max_{t\geq 0}\tilde{x}^N(t,0)>s$.

Combine the three points above, we obtain the desired results. □

C.3.2. Proof of Proposition 4. Case 1 follows from Lemma 4.1, 4.2, and (33)–(34). For Case 2, since $\max\limits_{t\geq 0}\tilde{x}^N(t,0)>s>\min_{t\geq 0}\tilde{x}^N(t,1)$, by Lemma 4.3 we have $\max_{t\geq 0}\tilde{x}^N(t,\beta)>s>\min_{t\geq 0}\tilde{x}^N(t,\beta)$, for any $\beta\in(0,1]$.

1. For $\beta=0$, $\bar{A}^N(0)<\bar{A}^F$: Since $\max_{t\geq 0}\tilde{x}^N(t,0)>s$, by Lemma 4.3.(b) we have $\tilde{x}^N(t,0)>\tilde{x}^F(t)$ $\forall t$ and $\max_{t\geq 0}\tilde{x}^F(t)>s$. Then, by (35) we have $\bar{A}^N(0)\leq\bar{A}^F$. Since $\min_{t\geq 0}\tilde{x}^N(t,0)<s$ implies $\min_{t\geq 0}\tilde{x}^F(t)<s$, there exists an interval $\eta_3\subseteq[0,d)$ such that for $t\in\eta_3$, $\tilde{x}^F(t)<s$ and $\tilde{x}^N(t,0)>\tilde{x}^F(t)$. Therefore, by (32), $\bar{A}^N(0)-\bar{A}^F=\frac{\mu}{d}\int_0^d(\tilde{x}^F(t)-(\tilde{x}^N(t,0)\wedge s))\,dt<0$.
2. For $\beta\geq 0.5$, $\bar{A}^N(\beta)>\bar{A}^F$: Since $\max_{t\geq 0}\tilde{x}^N(t,\beta)>s$, by Lemma 4.3.(a) we have $\tilde{x}^N(t,\beta)<\tilde{x}^F(t)$ $\forall t$. Then, by (36) we have $\bar{A}^F\leq\bar{A}^N(\beta)$. Since $\min_{t\geq 0}\tilde{x}^N(t,\beta)<s$, there exists an interval $\eta_3'\subseteq[0,d)$ such that for $t\in\eta_3'$, $\tilde{x}^N(t,\beta)<s$ and $\tilde{x}^N(t,\beta)<\tilde{x}^F$. Therefore, by (32), $\bar{A}^N(\beta)-\bar{A}^F=\frac{\mu}{d}\int_0^d((\tilde{x}^F(t)\wedge s)-\tilde{x}^N(t,\beta))\,dt>0$.
3. $\bar{A}^N(\beta)$ is increasing in β: Lemma 4.3 implies that $\tilde{x}^N(t,\beta)$ strictly decreases in β. For $\beta^h>\beta^l$, we have $\tilde{x}^N(t,\beta^h)<\tilde{x}^N(t,\beta^l)$. Since $\min_{t\geq 0}\tilde{x}^N(t,\beta)<s$, there exists an interval $\eta_3''\subseteq[0,d)$ such that for $t\in\eta_3''$, $\tilde{x}^N(t,\beta^h)<s$ and $\tilde{x}^N(t,\beta^h)<\tilde{x}^N(t,\beta^l)$. Therefore, by (32), $\bar{A}^N(\beta^h)-\bar{A}^N(\beta^l)=\frac{\mu}{d}\int_0^d((\tilde{x}^N(t,\beta^l)\wedge s)-\tilde{x}^N(t,\beta^h))\,dt>0$.

Combining the results above, we obtain the desired results. □

C.4. Proof for Section 6.5: Proposition 5.

In this case, the flow balance equations (13) and (14) specialize to

$$\bar{A}_1^I=\lambda_1-\mu_1(\bar{x}_1^I\wedge s), \tag{39}$$

$$\bar{A}_2^I=\lambda_2-\mu_2\left((s-\bar{x}_1^I)^+\wedge\bar{x}_2^I\right), \tag{40}$$

for $I\in\{F,N\}$, and (15)-(18) specialize to

$$\bar{A}_1^N:=\theta\left(\beta_1\frac{(\bar{x}_1^N+\bar{x}_2^N-s)^+}{s}\right)(\bar{x}_1^N-s)^+, \tag{41}$$

$$\bar{A}_1^F:=\int_0^{(\bar{x}_1^F-s)^+}\theta\left(u/s\right)du, \tag{42}$$

$$\bar{A}_2^N:=\theta\left(\beta_2\frac{(\bar{x}_1^N+\bar{x}_2^N-s)^+}{s}\right)(\bar{x}_2^N-(s-\bar{x}_1^N)^+)^+, \tag{43}$$

$$\bar{A}_2^F:=\int_{(\bar{x}_1^F-s)^+}^{(\bar{x}_1^F+\bar{x}_2^F-s)^+}\theta\left(u/s\right)du. \tag{44}$$

To compare the equilibrium numbers-in-system and system abandonment rates under different information levels, we first characterize these two metrics under each of the information levels in Lemma 5. The following lemma follows from (39)–(44).

Lemma 5. *For two-priority systems with stationary arrivals, the fluid approximation (7)–(8) has the following equilibrium point:*

1. *If $\rho_1 \leq 1$, then $\bar{x}_1^N(\boldsymbol{\beta}) = \bar{x}_1^F = s\rho_1$ and $\bar{A}_1^N(\boldsymbol{\beta}) = \bar{A}_1^F = 0$.*

   (i) If $\rho_1+\rho_2 \leq 1$, then $\bar{x}_2^N(\boldsymbol{\beta}) = \bar{x}_2^F = s\rho_2$ and $\bar{A}_2^N(\boldsymbol{\beta}) = \bar{A}_2^F = 0$.

   (ii) If $\rho_1+\rho_2 > 1$, then $\bar{x}_1^N(\boldsymbol{\beta}) + \bar{x}_2^N(\boldsymbol{\beta}) > s$, $\bar{x}_1^F + \bar{x}_2^F > s$, $\bar{A}_2^N(\boldsymbol{\beta}) = \bar{A}_2^F = s\mu_2(\rho_1+\rho_2-1)$, $\bar{x}_2^N(\boldsymbol{\beta})$ is the unique solution of

$$\mu_2(\rho_1+\rho_2-1) = \theta\left(\beta_2(\frac{x_2}{s}+\rho_1-1)\right)\left(\frac{x_2}{s}+\rho_1-1\right), \tag{45}$$

and $\bar{x}_2^F$ is the unique solution of

$$\mu_2(\rho_1+\rho_2-1) = \int_0^{\frac{x_2}{s}+\rho_1-1} \theta(u)du. \tag{46}$$

2. *If $\rho_1 > 1$, then $\bar{A}_1^N(\boldsymbol{\beta}) = \bar{A}_1^F = \lambda_1 - s\mu_1$ and $\bar{A}_2^N(\boldsymbol{\beta}) = \bar{A}_2^F = \lambda_2$. Further, $\bar{x}_1^N(\boldsymbol{\beta}) > s$, $\bar{x}_1^F > s$, $(\bar{x}_1^N(\boldsymbol{\beta}), \bar{x}_2^N(\boldsymbol{\beta}))$ is the unique solution of the following system of equations,*

$$\mu_1(\rho_1-1) = \theta\left(\beta_1\frac{x_1+x_2-s}{s}\right)\left(\frac{x_1}{s}-1\right), \tag{47}$$

$$\mu_2\rho_2 = \theta\left(\beta_2\frac{x_1+x_2-s}{s}\right)\frac{x_2}{s}, \tag{48}$$

and $(\bar{x}_1^F, \bar{x}_2^F)$ is the unique solution of the following system of equations

$$\mu_1(\rho_1-1) = \int_0^{\frac{x_1}{s}-1} \theta(u)du, \tag{49}$$

$$\mu_2\rho_2 = \int_{\frac{x_1}{s}-1}^{\frac{x_1}{s}+\frac{x_2}{s}-1} \theta(u)du. \tag{50}$$

Proof of Lemma 5. Divide both sides of (39) and (40) by $s\mu_1$ and $s\mu_2$, respectively, we can obtain that

$$\rho_1 = (\frac{\bar{x}_1^I}{s} \wedge 1) + \frac{\bar{A}_1^I}{s\mu_1}, \tag{51}$$

$$\rho_2 = ((1-\frac{\bar{x}_1^I}{s})^+ \wedge \frac{\bar{x}_2^I}{s}) + \frac{\bar{A}_2^I}{s\mu_2}. \tag{52}$$

We provide the proof for $I=F$, and a similar analysis yields the proof for $I=N$.

1. When $\rho_1 \leq 1$, we have $\bar{x}_1^F \leq s$. If on the contrary, $\bar{x}_1^F > s$, then

$$\rho_1 = (\frac{\bar{x}_1^F}{s} \wedge 1) + \frac{\bar{A}_1^F}{s\mu_1} \geq \frac{\bar{x}_1^F}{s} > 1,$$

which contradicts to $\rho_1 \leq 1$. By (42), when $\bar{x}_1^F \leq s$, $\bar{A}_1^F = 0$, thus $\bar{x}_1^F = s\rho_1$ by (51).

   (a) When $\rho_1+\rho_2 \leq 1$, we show $\bar{x}_1^F + \bar{x}_2^F \leq s$ by contradiction. If otherwise, $\bar{x}_1^F + \bar{x}_2^F > s$, then $\bar{x}_2^F > 0$, $\bar{A}_2^F > 0$, and by (52) we have

$$\rho_2 = (1-\rho_1) + \frac{\bar{A}_2^F}{s\mu_2} > 1-\rho_1,$$

which contradicts to $\rho_1+\rho_2 \leq 1$. Therefore, $\bar{x}_1^F + \bar{x}_2^F \leq s$, which implies that $\bar{A}_2^F = 0$, and $\bar{x}_2^F = s\mu_2$.

(b) When $\rho_1+\rho_2>1$, we show $\bar{x}_1^F+\bar{x}_2^F>s$ by contradiction. If otherwise, $\bar{x}_1^F+\bar{x}_2^F\le s$, then $\bar{x}_2^F\le s(1-\rho_1)<s\rho_2$ and $\bar{A}_2^F=0$. However, (52) implies that $\rho_2=\frac{\bar{x}_2^F}{s}$, which leads to a contradiction. Therefore, $\bar{x}_1^F+\bar{x}_2^F>s$, and $\bar{x}_2^F$ is the solution of

$$\rho_2=1-\rho_1+\frac{\bar{A}_2^F}{s\mu_2}.$$

Plug in (43)–(44) into the above equation, we can obtain the desired results.

2. When $\rho_1>1$, similar to the previous analysis, we can show that $\bar{x}_1^F>s$ by contradiction. Then, (51) and (52) can be simplified as

$$\rho_1=1+\frac{\bar{A}_1^F}{s\mu_1},\quad \rho_2=\frac{\bar{A}_2^F}{s\mu_2}.$$

Plug in (41)–(44) into the above equations, we can obtain the desired results. □

The average abandonment rate ranking immediately follows from Lemma 5. As for the number-in-system rankings, part of the results in Case 1 of Proposition 5 is proved by Lemma 5. We complete the proof for Case 1 of Proposition 5 by the continuity of $\bar{x}^N(\boldsymbol{\beta})$ in $\boldsymbol{\beta}$ (i.e., Proposition 10) and the following lemma.

LEMMA 6. *If $\rho_1\le 1$, $\rho_1+\rho_2>1$, then $\bar{x}_2^N(\boldsymbol{\beta})$ is non-increasing in β_2, $\bar{x}_2^N(\beta_1,1)<\bar{x}_2^F$, and $\bar{x}_2^N(\beta_1,0)>\bar{x}_2^F$.*

Proof of Lemma 6. When $\rho_1\le 1, \rho_1+\rho_2>1$, by Lemma 5, $\bar{x}_2^N(\boldsymbol{\beta})$ is the unique solution of equation (45). Differentiating both sides of (45) with respect to β_2 yields:

$$\frac{d\bar{x}_2^N(\boldsymbol{\beta})}{d\beta_2}=-\frac{s\theta'(\frac{\bar{x}_2^N(\boldsymbol{\beta})}{s}+\rho_1-1)(\frac{\bar{x}_2^N(\boldsymbol{\beta})}{s}+\rho_1-1)^2}{\theta(\beta_2(\frac{\bar{x}_2^N(\boldsymbol{\beta})}{s}+\rho_1-1))+\theta'(\beta_2(\frac{\bar{x}_2^N(\boldsymbol{\beta})}{s}+\rho_1-1))\beta_2(\frac{\bar{x}_2^N(\boldsymbol{\beta})}{s}+\rho_1-1)}<0.$$

Thus, $\bar{x}_2^N(\boldsymbol{\beta})$ is decreasing in β_2.

Next, we check the boundary case when $\beta_2=1$. Plugging $\beta_2=1$ in (45) we can obtain that:

$$\mu_2(\rho_1+\rho_2-1)=\theta(\frac{\bar{x}_2^N(1)}{s}+\rho_1-1)(\frac{\bar{x}_2^N(1)}{s}+\rho_1-1).$$

By Case 1.(ii) of Lemma 5, we have $\bar{x}_1^N(\beta_1,1)+\bar{x}_2^N(\beta_1,1)>s$. Then, $\bar{x}_2^N(\beta_1,1)<\bar{x}_2^F$ since if on the contrary $\bar{x}_2^N(\beta_1,1)\ge\bar{x}_2^F$, by (46) we have

$$\begin{aligned}\mu_2(\rho_1+\rho_2-1)&=\int_0^{\frac{\bar{x}_2^F}{s}+\rho_1-1}\theta(u)du\le\int_0^{\frac{\bar{x}_2^N(\beta_1,1)}{s}+\rho_1-1}\theta(u)du\\&<\theta\left(\frac{\bar{x}_2^N(\beta_1,1)}{s}+\rho_1-1\right)\left(\frac{\bar{x}_2^N(\beta_1,1)}{s}+\rho_1-1\right)=\mu_2(\rho_1+\rho_2-1).\end{aligned}$$

Thus, by contradiction, we have $\bar{x}_2^N(\beta_1,1)<\bar{x}_2^F$.

Lastly, we check the other boundary case when $\beta_2=0$. Plugging $\beta_2=0$ in (45) we obtain that

$$\mu_2(\rho_1+\rho_2-1)=\theta(0)(\frac{\bar{x}_2^N(\beta_1,0)}{s}+\rho_1-1).$$

Then, $\bar{x}_2^N(\beta_1,0)>\bar{x}_2^F$ since otherwise, by (46),

$$\mu_2(\rho_1+\rho_2-1)=\int_0^{\frac{\bar{x}_2^F}{s}+\rho_1-1}\theta(u)du>\theta(0)(\frac{\bar{x}_2^F}{s}+\rho_1-1)\ge\theta(0)(\frac{\bar{x}_2^N(\beta_1,0)}{s}+\rho_1-1)=\mu_2(\rho_1+\rho_2-1)$$

yields a contradiction. □

Next, we consider the case when $\rho_1>1$. Case 2 of Proposition 5 is deduced by Lemma 5 and the following lemma:

Lemma 7. *For two-priority systems with stationary arrivals, if $\rho_1 > 1$,*

1. *For HP customers: for any β_2, ρ_2, there exists $\beta_1^*(\beta_2) \in (0, 0.5)$ such that $\beta_1^*(\beta_2)$ is increasing in β_2, decreasing in ρ_2, and*
   (i) If $\beta_1 \in [\beta_1^*(\beta_2), 1]$, then $\bar{x}_1^N(\boldsymbol{\beta}) \le \bar{x}_1^F$.
   (ii) If $\beta_1 \in [0, \beta_1^*(\beta_2))$, then $\bar{x}_1^F < \bar{x}_1^N(\boldsymbol{\beta})$.
2. *For LP customers: for any β_1, ρ_2, there exists $\beta_2^*(\beta_1) \in (0, 1]$ such that $\beta_2^*(\beta_1)$ is increasing in β_1, decreasing in ρ_2, and*
   (i) If $\beta_2 \in [\beta_2^*(\beta_1), 1]$, then $\bar{x}_2^N(\boldsymbol{\beta}) \le \bar{x}_2^F$.
   (ii) If $\beta_2 \in [0, \beta_2^*(\beta_1))$, then $\bar{x}_2^F < \bar{x}_2^N(\boldsymbol{\beta})$.

Proof of Lemma 7. 1. We prove Part 1 of this lemma in two steps. First, we demonstrate the existence of such $\beta_1^*(\beta_2)$. Next, we prove its monotonicity with respect to the parameters.

Step 1: We prove the first step by showing that, for any fixed β_2, (a) $\bar{x}_1^N(\boldsymbol{\beta})$ is continuous and decreasing in β_1, (b) $\bar{x}_1^F < \bar{x}_1^N(0, \beta_2)$, and (c) $\bar{x}_1^F > \bar{x}_1^N(0.5, \beta_2)$.

(a) The continuity is given by Proposition 10. To show that $\bar{x}_1^N(\boldsymbol{\beta})$ is decreasing in β_1, differentiating both sides of equations (47) and (48) with respect to β_1 yields:

$$\begin{aligned}
&\beta_1\theta'\left(\beta_1\frac{\bar{x}_1^N(\boldsymbol{\beta})+\bar{x}_2^N(\boldsymbol{\beta})-s}{s}\right)\left(\frac{\bar{x}_1^N(\boldsymbol{\beta})}{s}-1\right)\frac{d(\bar{x}_1^N(\boldsymbol{\beta})+\bar{x}_2^N(\boldsymbol{\beta}))}{d\beta_1}+\theta\left(\beta_1\frac{\bar{x}_1^N(\boldsymbol{\beta})+\bar{x}_2^N(\boldsymbol{\beta})-s}{s}\right)\frac{d(\bar{x}_1^N(\boldsymbol{\beta}))}{d\beta_1}\\
&=-s\theta'\left(\beta_1\frac{\bar{x}_1^N(\boldsymbol{\beta})+\bar{x}_2^N(\boldsymbol{\beta})-s}{s}\right)\left(\frac{\bar{x}_1^N(\boldsymbol{\beta})}{s}-1\right)\left(\frac{\bar{x}_1^N(\boldsymbol{\beta})+\bar{x}_2^N(\boldsymbol{\beta})}{s}-1\right)<0. \qquad (53)\\
&\beta_2\theta'\left(\beta_2\frac{\bar{x}_1^N(\boldsymbol{\beta})+\bar{x}_2^N(\boldsymbol{\beta})-s}{s}\right)\frac{\bar{x}_2^N(\boldsymbol{\beta})}{s}\frac{d(\bar{x}_1^N(\boldsymbol{\beta})+\bar{x}_2^N(\boldsymbol{\beta}))}{d\beta_1}+\theta\left(\beta_2\frac{\bar{x}_1^N(\boldsymbol{\beta})+\bar{x}_2^N(\boldsymbol{\beta})-s}{s}\right)\frac{d(\bar{x}_2^N(\boldsymbol{\beta}))}{d\beta_1}=0. \qquad (54)
\end{aligned}$$

By (54), there are three possible cases: (1) $\frac{d(\bar{x}_1^N(\boldsymbol{\beta})+\bar{x}_2^N(\boldsymbol{\beta}))}{d\beta_1}=\frac{d(\bar{x}_2^N(\boldsymbol{\beta}))}{d\beta_1}=0$; (2) $\frac{d(\bar{x}_1^N(\boldsymbol{\beta})+\bar{x}_2^N(\boldsymbol{\beta}))}{d\beta_1}>0$, $\frac{d(\bar{x}_2^N(\boldsymbol{\beta}))}{d\beta_1}<0$, and thus $\frac{d(\bar{x}_1^N(\boldsymbol{\beta}))}{d\beta_1}>0$; or (3) $\frac{d(\bar{x}_1^N(\boldsymbol{\beta})+\bar{x}_2^N(\boldsymbol{\beta}))}{d\beta_1}<0$, $\frac{d(\bar{x}_2^N(\boldsymbol{\beta}))}{d\beta_1}>0$, and thus $\frac{d(\bar{x}_1^N(\boldsymbol{\beta}))}{d\beta_1}<0$. Both cases (1) and (2) are impossible as they contradict (53). Therefore, case (3) must be true. That is, $\bar{x}_1^N(\boldsymbol{\beta})$ is decreasing in β_1,$\bar{x}_2^N(\boldsymbol{\beta})$ is increasing in β_1, and $\bar{x}_1^N(\boldsymbol{\beta})+\bar{x}_2^N(\boldsymbol{\beta})$ is decreasing in β_1.

(b) Since $\bar{x}_1^N(\boldsymbol{\beta})>s$ and $\bar{x}_1^F>s$, plugging $\beta_1=0$ in (47) and together with (49) we have:

$$\mu_1(\rho_1-1)=\theta(0)(\frac{\bar{x}_1^N(0,\beta_2)}{s}-1)=\int_0^{\frac{\bar{x}_1^F}{s}-1}\theta(u)du>\theta(0)(\frac{\bar{x}_1^F}{s}-1)\Rightarrow\bar{x}_1^N(0,\beta_2)>\bar{x}_1^F.$$

(c) Plugging $\beta_1=0.5$ in (47) we have:

$$\theta\left(\frac{\bar{x}_1^N(0.5,\beta_2)-s}{2s}\right)(\frac{\bar{x}_1^N(0.5,\beta_2)}{s}-1)<\theta\left(\frac{\bar{x}_1^N(0.5,\beta_2)+\bar{x}_2^N(0.5,\beta_2)-s}{2s}\right)(\frac{\bar{x}_1^N(0.5,\beta_2)}{s}-1)=\mu_1(\rho_1-1).$$

Since θ is concave, then by (47) and (49) we have:

$$\theta(\frac{\bar{x}_1^F-s}{2s})(\frac{\bar{x}_1^F}{s}-1)>\int_0^{\frac{\bar{x}_1^F}{s}-1}\theta(u)du=\mu_1(\rho_1-1)>\theta\left(\frac{\bar{x}_1^N(0.5,\beta_2)-s}{2s}\right)(\frac{\bar{x}_1^N(0.5,\beta_2)}{s}-1).$$

Therefore, $\bar{x}_1^F>\bar{x}_1^N(0.5,\beta_2)$.

Combining (a)–(c), there exists a $\beta_1^*(\beta_2)\in(0,0.5)$ such that $\bar{x}_1^N(\boldsymbol{\beta})=\bar{x}_1^F$ when $\beta_1=\beta_1^*(\beta_2)$, $\bar{x}_1^N(\boldsymbol{\beta})<\bar{x}_1^F$ when $1\ge\beta_1>\beta_1^*(\beta_2)$ and $\bar{x}_1^F<\bar{x}_1^N(\boldsymbol{\beta})$ when $0<\beta_1<\beta_1^*(\beta_2)$.

Step 2: $\beta_1^*(\beta_2)$ is increasing in β_2 and decreasing in ρ_2. To facilitate the proof, we first show that $\bar{x}_1^N(\boldsymbol{\beta})$ is increasing in β_2, $\bar{x}_2^N(\boldsymbol{\beta})$ is decreasing in β_2, and $\bar{x}_1^N(\boldsymbol{\beta})+\bar{x}_2^N(\boldsymbol{\beta})$ is decreasing in β_2. Differentiating both sides of equations (47) and (48) with respect to β_2 yields:

$$\beta_1\theta'\left(\beta_1\frac{\bar{x}_1^N(\boldsymbol{\beta})+\bar{x}_2^N(\boldsymbol{\beta})-s}{s}\right)\frac{\bar{x}_1^N(\boldsymbol{\beta})-s}{s}\frac{d(\bar{x}_1^N(\boldsymbol{\beta})+\bar{x}_2^N(\boldsymbol{\beta}))}{d\beta_2}+\theta\left(\beta_1\frac{\bar{x}_1^N(\boldsymbol{\beta})+\bar{x}_2^N(\boldsymbol{\beta})-s}{s}\right)\frac{d(\bar{x}_1^N(\boldsymbol{\beta}))}{d\beta_2}=0. \tag{55}$$

$$\begin{aligned}&\beta_2\theta'\left(\beta_2\frac{\bar{x}_1^N(\boldsymbol{\beta})+\bar{x}_2^N(\boldsymbol{\beta})-s}{s}\right)\frac{\bar{x}_2^N(\boldsymbol{\beta})}{s}\frac{d(\bar{x}_1^N(\boldsymbol{\beta})+\bar{x}_2^N(\boldsymbol{\beta}))}{d\beta_2}+\theta\left(\beta_2\frac{\bar{x}_1^N(\boldsymbol{\beta})+\bar{x}_2^N(\boldsymbol{\beta})-s}{s}\right)\frac{d(\bar{x}_2^N(\boldsymbol{\beta}))}{d\beta_2}\\&=-s\theta'\left(\beta_2\frac{\bar{x}_1^N(\boldsymbol{\beta})+\bar{x}_2^N(\boldsymbol{\beta})-s}{s}\right)\frac{\bar{x}_2^N(\boldsymbol{\beta})}{s}\left(\frac{\bar{x}_1^N(\boldsymbol{\beta})+\bar{x}_2^N(\boldsymbol{\beta})-s}{s}\right)<0.\end{aligned} \tag{56}$$

By (55), there are three possible cases: (1) $\frac{d(\bar{x}_1^N(\boldsymbol{\beta})+\bar{x}_2^N(\boldsymbol{\beta}))}{d\beta_2}=\frac{d(\bar{x}_1^N(\boldsymbol{\beta}))}{d\beta_2}=0$; (2) $\frac{d(\bar{x}_1^N(\boldsymbol{\beta})+\bar{x}_2^N(\boldsymbol{\beta}))}{d\beta_2}>0$, $\frac{d(\bar{x}_1^N(\boldsymbol{\beta}))}{d\beta_2}<0$, and thus $\frac{d(\bar{x}_2^N(\boldsymbol{\beta}))}{d\beta_2}>0$; or (3) $\frac{d(\bar{x}_1^N(\boldsymbol{\beta})+\bar{x}_2^N(\boldsymbol{\beta}))}{d\beta_2}<0$, $\frac{d(\bar{x}_1^N(\boldsymbol{\beta}))}{d\beta_2}>0$, and thus $\frac{d(\bar{x}_2^N(\boldsymbol{\beta}))}{d\beta_2}<0$. Both cases (1) and (2) are impossible as they contradict (56). Therefore, case (3) must be true.

By (47) and (49), when $\beta_1=\beta_1^*(\beta_2)$, then $\bar{x}_1^N(\beta_1^*(\beta_2),\beta_2)=\bar{x}_1^F=:x_1^*$, and

$$\mu_1(\rho_1-1)=\theta\left(\beta_1^*(\beta_2)\frac{x_1^*+\bar{x}_2^N(\beta_1^*(\beta_2),\beta_2)-s}{s}\right)\left(\frac{x_1^*}{s}-1\right)=\int_0^{\frac{x_1^*}{s}-1}\theta(u)du.$$

Since $\bar{x}_2^N(\boldsymbol{\beta})$ is decreasing in β_2, an increase in β_2 leads to a decrease in $\bar{x}_2^N(\beta_1^*(\beta_2),\beta_2)$. As $x_1^*=\bar{x}_1^F$ is independent of $\boldsymbol{\beta}$ and remains unchanged, $\beta_1^*(\beta_2)$ must increase to maintain the above equation.

Similarly, when ρ_2 increases, $\bar{x}_1^N(\beta_1^*(\beta_2),\beta_2)=x_1^*$ remains unchanged, $\bar{x}_2^N(\beta_1^*(\beta_2),\beta_2)$ must increase to maintain (48), and thus $\beta_1^*(\beta_2)$ must decrease to maintain (47).

2. Similar to Part 1, we prove Part 2 in two steps.

Step 1: We show the existence of $\beta_2^*(\beta_1)$ by showing the following: for any fixed β_1, (a) $\bar{x}_2^N(\boldsymbol{\beta})$ is decreasing in β_2; (b) when $\beta_2=0$, $\bar{x}_2^F<\bar{x}_2^N(\beta_1,0)$; and (c) when $\beta_2=1$, (i) if $\beta_1\le\beta_1^*(\beta_2)$, then $\bar{x}_2^F>\bar{x}_2^N(\beta_1,1)$; (ii) if $\beta_1>\beta_1^*(\beta_2)$, then there exist a threshold ρ_2^* such that $\bar{x}_2^F<\bar{x}_2^N(\boldsymbol{\beta})$ if $\rho_2<\rho_2^*$ and $\bar{x}_2^F\ge\bar{x}_2^N(\boldsymbol{\beta})$ if $\rho_2\ge\rho_2^*$. Then, by the continuity of $\bar{x}_2^N(\boldsymbol{\beta})$ on β_2, such $\beta_2^*(\beta_1)$ exists and when $\beta_1>\beta_1^*(\beta_2)$ and $\rho_2<\rho_2^*$, $\beta_2^*(\beta_1)=1$; otherwise, $0<\beta_2^*(\beta_1)<1$.

Since part (a) is already proven in step 2 of part 1, our focus shifts to parts (b) and (c).

(b) When $\beta_2=0$, by (48) and (50) we have

$$\mu_2\rho_2=\theta(0)\frac{\bar{x}_2^N(\beta_1,0)}{s}=\int_{\frac{\bar{x}_1^F}{s}-1}^{\frac{\bar{x}_1^F}{s}+\frac{\bar{x}_2^F}{s}-1}\theta(u)du>\int_{\frac{\bar{x}_1^F}{s}-1}^{\frac{\bar{x}_1^F}{s}+\frac{\bar{x}_2^F}{s}-1}\theta(0)du=\theta(0)\frac{\bar{x}_2^F}{s}.$$

Thus, $\bar{x}_2^N(\beta_1,0)>\bar{x}_2^F$.

(c.i) When $\beta_2=1$, if $\beta_1\le\beta_1^*(\beta_2)$, then by part 1 we have $\bar{x}_1^N(\boldsymbol{\beta})\ge\bar{x}_1^F$. By by (48) and (50) we have

$$\begin{aligned}\mu_2\rho_2&=\theta\left(\frac{\bar{x}_1^N(\beta_1,1)+\bar{x}_2^N(\beta_1,1)-s}{s}\right)\frac{\bar{x}_2^N(\beta_1,1)}{s}=\int_{\frac{\bar{x}_1^F}{s}-1}^{\frac{\bar{x}_1^F}{s}+\frac{\bar{x}_2^F}{s}-1}\theta(u)du\\&<\theta\left(\frac{\bar{x}_1^F+\bar{x}_2^F-s}{s}\right)\frac{\bar{x}_2^F}{s}\le\theta\left(\frac{\bar{x}_1^N(\beta_1,1)+\bar{x}_2^F-s}{s}\right)\frac{\bar{x}_2^F}{s}.\end{aligned}$$

Therefore, $\bar{x}_2^N(\beta_1,1)<\bar{x}_2^F$ since θ is increasing.

(c.ii) When $\beta_2 = 1$, if $\beta_1 > \beta_1^*(\beta_2)$, then by part 1 we have $\bar{x}_1^N(\beta_1,1) < \bar{x}_1^F$. Let $\Delta = \bar{x}_1^F - \bar{x}_1^N(\beta_1,1) > 0$. We will show that there exist a threshold ρ_2^* such that $\bar{x}_2^F < \bar{x}_2^N(\boldsymbol{\beta})$ if $\rho_2 < \rho_2^*$ and $\bar{x}_2^F \geq \bar{x}_2^N(\boldsymbol{\beta})$ if $\rho_2 \geq \rho_2^*$. Let

$$g(x) = \theta(\frac{\bar{x}_1^N(x) + x - s}{s})x - \int_{\bar{x}_1^F - s}^{\bar{x}_1^F + x - s} \theta(\frac{u}{s})du.$$

We first show that $g(x)$ has a unique positive root in two steps. In step A, we define two alternative systems of equations with solutions $(\underline{x}_{1,m}^N, \underline{x}_{2,m}^N)$, $(\bar{x}_{1,m}^N, \bar{x}_{2,m}^N)$ such that for any m, $\underline{x}_{1,m}^N$ is independent of $\underline{x}_{2,m}^N$, $\bar{x}_{1,m}^N$ is independent of $\bar{x}_{2,m}^N$, and $\underline{x}_{1,m}^N \uparrow \bar{x}_1^N(\beta_1,1), \bar{x}_{1,m}^N \downarrow \bar{x}_1^N(\beta_1,1)$ as $m \to \infty$. In step B, we identify functions $\bar{g}_m$ and $\underline{g}_m$ such that $\bar{g}_m$ and $\underline{g}_m$ have unique positive roots for $\forall m > 0$, and $\underline{g}_m(x) \uparrow g(x), \bar{g}_m(x) \downarrow g(x)$ as $m \to \infty$.

Step A: Define alternative systems of equations, indexed by m where $m \to \infty$, as follows. Let $\bar{x}_{1,m}^N$ and $\bar{x}_{2,m}^N$ denote the solutions to the following system of equations:

$$\mu_1(\rho_1 - 1) = \theta(\beta_1 \frac{x_{1,m}^N + \zeta_m - s}{s})(\frac{x_{1,m}^N}{s} - 1), \tag{57}$$
$$\mu_2\rho_2 = \theta(\frac{x_{1,m}^N + x_{2,m}^N - s}{s})\frac{x_{2,m}^N}{s}, \tag{58}$$

where we let $\{\zeta_m, m \geq 0\}$ be a sequence converging from below to $\bar{x}_2^N(\beta_1,1)$ i.e., $\zeta_m \leq \bar{x}_2^N(\beta_1,1)$ and $\zeta_m \uparrow \bar{x}_2^N(\beta_1,1)$. Similarly, let $\underline{x}_{1,m}^N$ and $\underline{x}_{2,m}^N$ denote the solutions to the following system of equations:

$$\mu_1(\rho_1 - 1) = \theta(\beta_1 \frac{x_{1,m}^N + \xi_m - s}{s})(\frac{x_{1,m}^N}{s} - 1), \tag{59}$$
$$\mu_2\rho_2 = \theta(\frac{x_{1,m}^N + x_{2,m}^N - s}{s})\frac{x_{2,m}^N}{s}. \tag{60}$$

where we let $\{\xi_m, m \geq 0\}$ be a sequence converging from above to $\bar{x}_2^N(\beta_1,1)$ i.e., $\xi_m \geq \bar{x}_2^N(\beta_1,1)$ and $\xi_m \downarrow \bar{x}_2^N(\beta_1,1)$. Then, by comparing equations (47), (57), and (59), and the monotonicity of θ, we must have $\underline{x}_{1,m}^N \leq \bar{x}_1^N(\beta_1,1) \leq \bar{x}_{1,m}^N$ for every $m \geq 0$. Moreover, by the continuity and monotonicity of θ we can obtain that $\underline{x}_{1,m}^N \to \bar{x}_1^N(\beta_1,1)$ and $\bar{x}_{1,m}^N \to \bar{x}_1^N(\beta_1,1)$ as $m \to \infty$. To see this, suppose on the contrary that there exists $\epsilon > 0$ such that $\bar{x}_1^N(\beta_1,1) - \underline{x}_{1,m}^N > \epsilon$ for $\forall m > 0$. Then, since $\xi_m \downarrow \bar{x}_2^N(\beta_1,1)$, we have $\xi_m - \bar{x}_2^N(\beta_1,1) < \frac{\epsilon}{2}$ for m sufficiently large. Thus, $\underline{x}_{1,m}^N + \xi_m - \bar{x}_1^N(\beta_1,1) - \bar{x}_2^N(\beta_1,1) < -\frac{\epsilon}{2} < 0$ for large enough m, and

$$\mu_1(\rho_1 - 1) = \theta(\beta_1 \frac{\underline{x}_{1,m}^N + \xi_m - s}{s})(\frac{\underline{x}_{1,m}^N}{s} - 1) < \theta(\beta_1 \frac{\bar{x}_1^N(\beta_1,1) + \bar{x}_2^N(\beta_1,1) - s}{s})(\frac{\bar{x}_1^N(\beta_1,1)}{s} - 1) = \mu_1(\rho_1 - 1),$$

which yields contradiction. Therefore, $\underline{x}_{1,m}^N \to \bar{x}_1^N(\beta_1,1)$ as $m \to \infty$. Similarly, we can obtain that $\bar{x}_{1,m}^N \to \bar{x}_1^N(\beta_1,1)$ as $m \to \infty$.

Step B: Define the functions $\bar{g}_m$ and $\underline{g}_m$ as follows:

$$\bar{g}_m(x) = \theta(\frac{\bar{x}_{1,m}^N + x - s}{s})x - \int_{\bar{x}_1^F - s}^{\bar{x}_1^F + x - s} \theta(\frac{u}{s})du,$$

and

$$\underline{g}_m(x) = \theta(\frac{\underline{x}_{1,m}^N + x - s}{s})x - \int_{\bar{x}_1^F - s}^{\bar{x}_1^F + x - s} \theta(\frac{u}{s})du.$$

It is clear that $\underline{g}_m(x) \leq g(x) \leq \bar{g}_m(x)$ for $\forall m > 0, x \geq 0$. Furthermore, $\underline{g}_m(x) \to g(x)$ and $\bar{g}_m(x) \to g(x)$ as $m \to \infty$.

Now we show that $\underline{g}_m(x)$ has a unique positive root, denoted as $\underline{x}^2_{2,m}$, and $\underline{g}_m(x) < 0$ for $0 < x < \underline{x}^2_{2,m}$, $\underline{g}_m(x) > 0$ for $x > \underline{x}^2_{2,m}$.

Recall that $\Delta = \bar{x}^F_1 - \bar{x}^N_1(\beta_1, 1) > 0$, then $\Delta_1 := \bar{x}^F_1 - \underline{x}^N_{1,m} > \Delta > 0$. Note that, $\underline{g}_m(0) = 0$. When $0 < x \leq \Delta_1$,

$$\theta(\frac{\underline{x}^N_{1,m} + x - s}{s})x \leq \theta(\frac{\bar{x}^F_1 - s}{s})x < \int_{\bar{x}^F_1 - s}^{\bar{x}^F_1 + x - s} \theta(\frac{u}{s})du \Rightarrow \underline{g}_m(x) < 0.$$

When $x > \Delta_1$, since $\theta(\cdot)$ is increasing and concave, we have

$$\begin{aligned} \underline{g}'_m(x) &= \frac{x}{s}\theta'(\frac{\underline{x}^N_{1,m} + x - s}{s}) + \theta(\frac{\underline{x}^N_{1,m} + x - s}{s}) - \theta(\frac{\bar{x}^F_1 + x - s}{s}) \\ &> \frac{x}{s}\theta'(\frac{\underline{x}^N_{1,m} + x - s}{s}) - \frac{\Delta_1}{s}\theta'(\frac{\underline{x}^N_{1,m} + x - s}{s}) \geq 0. \end{aligned}$$

Thus $\underline{g}_m(x)$ is increasing when $x > \Delta_1$. Moreover,

$$\underline{g}_m(2\Delta_1) = \theta(\frac{\bar{x}^F_1 + \Delta_1 - s}{s})2\Delta_1 - \int_{\bar{x}^F_1 - s}^{\bar{x}^F_1 + 2\Delta_1 - s} \theta(\frac{u}{s})du > 0.$$

Therefore, there exists a unique $\underline{x}^2_{2,m} > 0$ such that $\underline{g}_m(\underline{x}^2_{2,m}) = 0$, and when $0 < x < \underline{x}^2_{2,m}$, $\underline{g}_m(x) < 0$; when $x > \underline{x}^2_{2,m}$, $\underline{g}_m(x) > 0$.

By a similar analysis, we can show that $\bar{g}_m(x)$ also has a unique positive root (denoted as $\bar{x}^2_{2,m}$), and $\bar{g}_m(x) < 0$ for $0 < x < \bar{x}^2_{2,m}$, $\bar{g}_m(x) > 0$ for $x > \bar{x}^2_{2,m}$. (Note that since $\bar{x}^N_{1,m} \to \bar{x}^N_1(\beta_1, 1)$ as $m \to \infty$, there exist $\bar{m}$ such that when $m > \bar{m}$, $\bar{x}^N_{1,m} - \bar{x}^N_1(\beta_1, 1) < \frac{1}{2}\Delta$. Then, $\Delta_2 := \bar{x}^F_1 - \bar{x}^N_{1,m} > \frac{1}{2}\Delta > 0$ when $m > \bar{m}$, and then the result follows from a similar analysis.)

Both $\underline{x}^2_{2,m}$ and $\bar{x}^2_{2,m}$ converge to some point $x^2_2 > 0$ by continuity of θ. Moreover, x^2_2 is the unique positive root of $g(x)$. First, we must have $g(x^2_2) = 0$. Otherwise, if $g(x^2_2) = \delta > 0$, then since $\bar{x}^2_{2,m} \to x^2_2$, we have $\bar{g}_m(x^2_2) - \bar{g}_m(\bar{x}^2_{2,m}) = \bar{g}_m(x^2_2) < \delta$ for m large enough, which contradicts to the fact that $g(x^2_2) \leq \bar{g}_m(x^2_2)$. Also, x^2_2 is the unique positive root. Otherwise, if there exists another positive root of $g(x)$, denoted as $\hat{x}^2_2 \neq x^2_2$. Then, we must have either $\bar{g}_m(\hat{x}^2_2) > 0, \underline{g}_m(\hat{x}^2_2) > 0$ or $\bar{g}_m(\hat{x}^2_2) < 0, \underline{g}_m(\hat{x}^2_2) < 0$ for m sufficiently large, and thus contradicts with $\underline{g}_m(x) \leq g(x) \leq \bar{g}_m(x)$. In conclusion, x^2_2 is the unique positive root of $g(x)$. Also, we can show that $g(x) < 0$ for $0 < x < x^2_2$, and $g(x) > 0$ for $x > x^2_2$. Let

$$\rho^*_2 := \frac{1}{\mu_2}\int_{\frac{\bar{x}^F_1}{s} - 1}^{\frac{\bar{x}^F_1}{s} + \frac{x^2_2}{s} - 1} \theta(u)du,$$

then when $\rho_2 < \rho^*_2$, we have $\bar{x}^F_2 < x^2_2$, $g(\bar{x}^F_2) < 0$; when $\rho_2 \geq \rho^*_2$, we have $\bar{x}^F_2 \geq x^2_2$, $g(\bar{x}^F_2) \geq 0$. Then when $\rho_2 < \rho^*_2$, we must have $\bar{x}^F_2 < \bar{x}^N_2(\beta_1, 1)$ since otherwise, if $\bar{x}^F_2 \geq \bar{x}^N_2(\beta_1, 1)$,

$$\mu_2\rho_2 = \theta\left(\frac{\bar{x}^N_1(\beta_1, 1) + \bar{x}^N_2(\beta_1, 1) - s}{s}\right)\frac{\bar{x}^N_2(\beta_1, 1)}{s} \leq \theta\left(\frac{\bar{x}^N_1(\beta_1, 1) + \bar{x}^F_2 - s}{s}\right)\frac{\bar{x}^F_2}{s} < \int_{\frac{\bar{x}^F_1}{s} - 1}^{\frac{\bar{x}^F_1}{s} + \frac{\bar{x}^F_2}{s} - 1} \theta(u)du = \mu_2\rho_2$$

yields a contradiction. Thus, when $\rho_2 < \rho^*_2$, $\bar{x}^F_2 < \bar{x}^N_2(\beta_1, 1)$. Similarly, when $\rho_2 \geq \rho^*_2$, $\bar{x}^F_2 \geq \bar{x}^N_2(\beta_1, 1)$.

Step 2: the proof is similar to to the method used in Step 2 for Case 1; therefore, we omit it here for brevity.

□

C.5. Proofs for Section 6.6

In this section, we provide proofs for the comparison results for systems under two-class and non-stationary periodic arrivals.

For the ease of the analysis for the proofs in this section, we write down the net flow rate functions as follows:

$$
\begin{aligned}
f_1^N(t,x,\boldsymbol{\beta}) &= \lambda_1(t) - \mu_1(x_1(t)\wedge s) - \theta\left(\beta_1 \frac{(x_1(t)+x_2(t)-s)^+}{s}\right)(x_1(t)-s)^+,\\
f_1^F(t,x) &= \lambda_1(t) - \mu_1(x_1(t)\wedge s) - \int_0^{(x_1(t)-s)^+} \theta(u/s)\,du,\\
f_2^N(t,x,\boldsymbol{\beta}) &= \lambda_2(t) - \mu_2\left((s-x_1(t))^+ \wedge x_2(t)\right) - \theta\left(\beta_2 \frac{(x_1(t)+x_2(t)-s)^+}{s}\right)(x_2(t)-(s-x_1(t))^+)^+,\\
f_2^F(t,x) &= \lambda_2(t) - \mu_2\left((s-x_1(t))^+ \wedge x_2(t)\right) - \int_{(x_1(t)-s)^+}^{(x_1(t)+x_2(t)-s)^+} \theta(u/s)\,du.
\end{aligned}
$$

Also, by the balance equations (13)–(18) we have

$$
0 = \int_0^d f_k^N(t,\tilde{x}^N,\boldsymbol{\beta})dt = \int_0^d f_k^F(t,\tilde{x}^F)dt, \text{ for } k=1,2. \tag{61}
$$

In Sections C.5.1–C.5.2, we present proofs of Propositions 6–7, respectively. Proposition 8 follows from Proposition 7, and its proof is similar to the proof of Proposition 4, so we omit it here.

C.5.1. Proof of Proposition 6. The following lemma 8 and the equations (13) and (14) imply the rankings of the equilibrium average number-in-system and average abandonment rates shown in Proposition 6.

LEMMA 8. *For two-priority systems with non-stationary periodic arrivals and uniformly underloaded HP class (i.e., $\bar{\rho}_1 \le 1$), information has the following effects on the numbers-in-system:*

1. *For HP customers: $\tilde{x}_1^N(t,\boldsymbol{\beta}) = \tilde{x}_1^F(t) \le s$ for all $t \ge 0$.*
2. *For LP customers: $\tilde{x}_2^N(t,\boldsymbol{\beta})$ is continuously non-increasing in β_2, with strict decreasing if $\max_{t\ge 0}(\tilde{x}_1^N(t,(\beta_1,0)) + \tilde{x}_2^N(t,(\beta_1,0))) > s$ and:*
   - *(i) If $\beta_2 \ge 0.5$, then $\tilde{x}_2^N(t,\boldsymbol{\beta}) \le \tilde{x}_2^F(t)$ for all $t \ge 0$;*
     - i. *If $\max_{t\ge 0}(\tilde{x}_1^N(t,\boldsymbol{\beta}) + \tilde{x}_2^N(t,\boldsymbol{\beta})) > s$, then the inequality is strict for all t,*
     - ii. *If $\max_{t\ge 0}(\tilde{x}_1^F(t) + \tilde{x}_2^F(t,)) > s$, then $\max_{t\ge 0}(\tilde{x}_1^N(t,\boldsymbol{\beta}) + \tilde{x}_2^N(t,\boldsymbol{\beta})) > s$.*
   - *(ii) If $\beta_2 = 0$, then $\tilde{x}_2^N(t,\boldsymbol{\beta}) \ge \tilde{x}_2^F(t)$ for all $t \ge 0$;*
     - i. *If $\max_{t\ge 0}(\tilde{x}_1^F(t) + \tilde{x}_2^F(t)) > s$, then the inequality is strict for all t,*
     - ii. *If $\max_{t\ge 0}(\tilde{x}_1^N(t,(\beta_1,0)) + \tilde{x}_2^N(t,(\beta_1,0))) > s$, then $\max_{t\ge 0}(\tilde{x}_1^F(t) + \tilde{x}_2^F(t)) > s$.*

Proof of Lemma 8. When $\bar{\rho}_1 \le 1$,

1. For HP customers, the proof of $\tilde{x}_1^N(t,\boldsymbol{\beta}) = \tilde{x}_1^F(t) \le s$ is similar to that of case 1 in Lemma 4, so we omit it here. Moreover, combining with (13) and (15)–(16), we have $\bar{A}_1^N(\boldsymbol{\beta}) = \bar{A}_1^F = 0$ and $\int_0^d \lambda_1(t)dt = \mu_1 \int_0^d \tilde{x}_1^N(t,\boldsymbol{\beta})dt = \mu_1 \int_0^d \tilde{x}_1^F(t)dt$.

When $\tilde{x}_1^N(t,\boldsymbol{\beta}) = \tilde{x}_1^F(t) \le s$, then $f_1^N(t,\tilde{x}^N,\boldsymbol{\beta}) = f_1^F(t,\tilde{x}^F) = \lambda_1(t) - \mu_1(\tilde{x}_1^F(t)\wedge s)$. That is, the solutions of $\tilde{x}_1^N(t,\boldsymbol{\beta})$, $\tilde{x}_1^F(t)$ are given by solving (7) and independent of $\boldsymbol{\beta}$, $\tilde{x}_2^N(t,\boldsymbol{\beta})$, and $\tilde{x}_2^F(t)$.

2. For LP customers, we compare the two metrics between N and F by Lemmas 1 and 2. Note that, Lemmas 1 and 2 only apply to one dimensional initial value problems. Since when $\bar{\rho}_1 \le 1$, $\tilde{x}_1^N(t,\boldsymbol{\beta})$ is independent of $\tilde{x}_2^N(t,\boldsymbol{\beta})$ and $\tilde{x}_1^N(t,\boldsymbol{\beta}) = \tilde{x}_1^F(t)$. We can convert the two dimensional initial value problems into one dimensional ones for LP customers by rewriting $f_2^N(t,x,\boldsymbol{\beta})$ and $f_2^F(t,x)$ as a function of $x_2(t)$ with $x_1(t) = \tilde{x}_1^F(t)$ fixed and given by (7). Let $y(t) = \tilde{x}_2^N(t,\boldsymbol{\beta})$, $z(t) = \tilde{x}_2^F(t)$, $f(t,y) = f_2^N(t,(\tilde{x}_1^N,x_2),\boldsymbol{\beta}) =: f_2^N(t,x_2,\boldsymbol{\beta})$, and $g(t,z) = f_2^F(t,(\tilde{x}_1^F,x_2)) := f_2^F(t,x_2)$. The proof of $\tilde{x}_2^N(t,,\boldsymbol{\beta})$ being continuously decreasing in β_2 is similar to that of case 3 in Lemma 4, so we omit it here. Note that,

$$f_2^N(t,x_2,\boldsymbol{\beta}) - f_2^F(t,x_2) = \int_0^{(\tilde{x}_1^F(t)+x_2(t)-s)^+} \theta\left(u/s\right)du - \theta\left(\beta_2 \frac{(\tilde{x}_1^F(t)+x_2(t)-s)^+}{s}\right)(\tilde{x}_1^F(t)+x_2(t)-s)^+.$$

Thus, $f_2^N(t,x_2,\boldsymbol{\beta}) - f_2^F(t,x_2) \le 0$ when $\beta_2 \ge 0.5$ and $f_2^N(t,x_2,\boldsymbol{\beta}) - f_2^F(t,x_2) \ge 0$ when $\beta_2 = 0$. Using a similar argument as in the proof of case 3 of Lemma 4, we can obtain the desired results. $\square$

C.5.2. Proof of Proposition 7 Denote $\varphi_2(l(t))$ as the unique solution of (7) under N with $x_2(t) = l(t)$ as given, for any non-negative periodic function $l(t)$. That is, $\varphi_2(l)$ solves the following ODE:

$$\dot{x}_1(t) = \lambda_1(t) - \mu_1(x_1(t)\wedge s) - \theta\left(\beta_1 \frac{(x_1(t)+l(t)-s)^+}{s}\right)(x_1(t)-s)^+.$$

Then, to facilitate the proof of Proposition 7, we introduce the following lemma. The proof of this lemma is similar to the proof of Lemma 7, thus we omit it here.

Lemma 9. 1. *$\varphi_2(l)$ is non-increasing in $l(t)$, with strict decreasing if $\max_{t\ge0}\varphi_2(l) > s$.*

2. *If $\max_{t\ge0}\varphi_2(l_0) > s$, for some $l_0(t)$, then $\max_{t\ge0}\varphi_2(l) > s$, for any $l(t)$.*
3. *When $\beta_1 \ge 0.5$, $\varphi_2(t,\boldsymbol{\beta},0) := \varphi_2(0) \le \tilde{x}_1^F(t)$, $\forall t$; if $\max_{t\ge0}\tilde{x}_1^F(t) > s$, then the inequality is strict for all t.*

Proof of Proposition 7 When $\bar{\rho}_1 > 1$,

1. When $\beta_1 = 0$, for HP customers, $\tilde{x}_1^N(t,\boldsymbol{\beta})$ and $f_1^N(t,x,\boldsymbol{\beta})$ are independent of $\tilde{x}_2^N(t,\boldsymbol{\beta})$ and

$$f_1^N(t,x_1,(0,\beta_2)) = \lambda_1(t) - \mu_1(x_1(t)\wedge s) - \theta(0)(x_1(t)-s)^+,$$
$$f_1^F(t,x_1) = \lambda_1(t) - \mu_1(x_1(t)\wedge s) - \int_0^{(x_1(t)-s)^+}\theta\left(u/s\right)du.$$

It is clear that $f_1^N(t,x_1,(0,\beta_2)) \ge f_1^F(t,x_1)$ for any $t \ge 0$. By a similar argument as we prove case 3 of Lemma 4 by applying Lemmas 1 and 2, we can obtain that $\tilde{x}_1^N(t,\boldsymbol{\beta}) \ge \tilde{x}_1^F(t)$ $\forall t$, with strict inequality for all t if $\max_{t\ge0}\tilde{x}_1^N(t,\boldsymbol{\beta}) > s$.

For LP customers: since $\tilde{x}_1^N(t,(0,\beta_2))$ and $\tilde{x}_1^F(t)$ are independent of $\tilde{x}_2^N(t,(0,\beta_2))$ and $\tilde{x}_2^F(t)$, we can rewrite $f_2^N(t,x,(0,\beta_2))$ and $f_2^F(t,x)$ as a function of $x_2(t)$ as follows:

$$\begin{aligned} f_2^N(t,x_2,(0,\beta_2)) =& \lambda_2(t) - \mu_2\left((s-\tilde{x}_1^N(t,(0,\beta_2)))^+ \wedge x_2(t)\right) \\ & - \theta\left(\beta_2\frac{(\tilde{x}_1^N(t,(0,\beta_2))+x_2(t)-s)^+}{s}\right)(x_2(t)-(s-\tilde{x}_1^N(t,(0,\beta_2)))^+)^+, \\ f_2^F(t,x_2) =& \lambda_2(t) - \mu_2\left((s-\tilde{x}_1^F(t))^+ \wedge x_2(t)\right) - \int_{(\tilde{x}_1^F(t)-s)^+}^{(\tilde{x}_1^F(t)+x_2(t)-s)^+}\theta\left(u/s\right)du. \end{aligned}$$

Define the difference between these two net flow rate function as follows:

$$\begin{aligned} &\Delta f_2(t,x_2,(0,\beta_2)) := f_2^N(t,x_2,(0,\beta_2)) - f_2^F(t,x_2) \\ &= \mu_2\left((s-\tilde{x}_1^F(t))^+ \wedge x_2(t)\right) - \mu_2\left((s-\tilde{x}_1^N(t,(0,\beta_2)))^+ \wedge x_2(t)\right) \\ &+ \int_{(\tilde{x}_1^F(t)-s)^+}^{(\tilde{x}_1^F(t)+x_2(t)-s)^+}\theta\left(u/s\right)du - \theta\left(\beta_2\frac{(\tilde{x}_1^N(t,(0,\beta_2))+x_2(t)-s)^+}{s}\right)(x_2(t)-(s-\tilde{x}_1^N(t,(0,\beta_2)))^+)^+. \end{aligned}$$

(a) When $\beta_2 = 0$, we show that if $\min_{t\geq 0} \tilde{x}_1^F(t) \geq s$, then $\tilde{x}_2^F(t) < \tilde{x}_2^N(t,(0,0)), \forall t$. Note that $\min_{t\geq 0} \tilde{x}_1^F(t) \geq s$ and the HP ranking result imply that $\tilde{x}_1^N(t,(0,0)) > \tilde{x}_1^F(t) \geq s$, for all t. Therefore, for any $x_2(t) > 0$, $\Delta f_2(t, x_2, (0, \beta_2))$ can be simplified as follows:

$$\Delta f_2(t, x_2, (0,0)) = \int_{\tilde{x}_1^F(t)-s}^{\tilde{x}_1^F(t)+x_2(t)-s} \theta\left(u/s\right) du - \theta(0) x_2(t) > 0.$$

Using a similar argument as we prove Lemma 4, we can find $t_4 \in [0, d)$ such that $\tilde{x}_2^F(t_4) < \tilde{x}_2^N(t_4,(0,0))$ by contradiction via (61) with $k = 2$. Apply Lemma 2 with $y(t) = \tilde{x}_2^N(t,(0,0)), z(t) = \tilde{x}_2^F(t)$, $f(t,y) = f_2^N(t, x_2, (0,0))$, and $g(t,z) = f_2^F(t, x_2)$, we obtain the desired result.

(b) When $\beta_2 = 1$, we show that if $\max_{t\geq 0} \tilde{x}_1^F(t) > s$ and $\mu_2 \leq \theta(0)$, then $\tilde{x}_2^F(t) > \tilde{x}_2^N(t,(0,1)), \forall t$. We first show that $\Delta f_2(t, x_2, (0,1)) \leq 0$, for all t. When $\max_{t\geq 0} \tilde{x}_1^F(t) > s$, we have $\tilde{x}_1^F(t) < \tilde{x}_1^N(t,(0,1))$ for all t. For $x_2(t) > 0$,

i. If $\tilde{x}_1^F(t) < \tilde{x}_1^N(t,(0,1)) \leq s$,

A. If $x_2(t) \leq s - \tilde{x}_1^N(t,(0,1))$, then the case is trivial with $\Delta f_2(t, x_2, (0,1)) = 0$.

B. If $s - \tilde{x}_1^N(t,(0,1)) < x_2(t) \leq s - \tilde{x}_1^F(t)$, then when $\mu_2 \leq \theta(0)$,

$$\begin{aligned}
\Delta f_2(t, x_2, (0,1)) &= \mu_2(\tilde{x}_1^N(t,(0,1)) + x_2(t) - s) - \theta\left(\frac{\tilde{x}_1^N(t,(0,1)) + x_2(t) - s}{s}\right)(\tilde{x}_1^N(t,(0,1)) + x_2(t) - s) \\
&< \mu_2(\tilde{x}_1^N(t,(0,1)) + x_2(t) - s) - \theta(0)(\tilde{x}_1^N(t,(0,1)) + x_2(t) - s) \leq 0.
\end{aligned}$$

C. If $x_2(t) > s - \tilde{x}_1^F(t)$, then when $\mu_2 \leq \theta(0)$,

$$\begin{aligned}
\Delta f_2(t, x_2, (0,1)) =& \mu_2(\tilde{x}_1^N(t,(0,1)) - \tilde{x}_1^F(t)) + \int_0^{\tilde{x}_1^F(t)+x_2(t)-s} \theta\left(u/s\right) du \\
&- \theta\left(\frac{\tilde{x}_1^N(t,(0,1)) + x_2(t) - s}{s}\right)(\tilde{x}_1^N(t,(0,1)) + x_2(t) - s) \\
\leq& \mu_2(\tilde{x}_1^N(t,(0,1)) - \tilde{x}_1^F(t)) + \int_0^{\tilde{x}_1^F(t)+x_2(t)-s} \theta\left(u/s\right) du - \int_0^{\tilde{x}_1^N(t,(0,1))+x_2(t)-s} \theta\left(u/s\right) du \\
=& \mu_2(\tilde{x}_1^N(t,(0,1)) - \tilde{x}_1^F(t)) - \int_{\tilde{x}_1^F(t)+x_2(t)-s}^{\tilde{x}_1^N(t,(0,1))+x_2(t)-s} \theta\left(u/s\right) du \\
<& \mu_2(\tilde{x}_1^N(t,(0,1)) - \tilde{x}_1^F(t)) - \int_{\tilde{x}_1^F(t)+x_2(t)-s}^{\tilde{x}_1^N(t,(0,1))+x_2(t)-s} \theta\left(0\right) du = (\mu_2 - \theta(0))(\tilde{x}_1^N(t,(0,1)) - \tilde{x}_1^F(t)) \leq 0.
\end{aligned}$$

ii. When $\tilde{x}_1^F(t) \leq s < \tilde{x}_1^N(t,(0,1))$,

A. If $x_2(t) \leq s - \tilde{x}_1^F(t)$, then when $\mu_2 \leq \theta(0)$,

$$\Delta f_2(t, x_2, (0,1)) = \mu_2 x_2(t) - \theta\left(\frac{\tilde{x}_1^N(t,(0,1)) + x_2(t) - s}{s}\right) x_2(t) < (\mu_2 - \theta(0)) x_2(t) \leq 0.$$

B. If $x_2(t) > s - \tilde{x}_1^F(t)$, then when $\mu_2 \leq \theta(0)$,

$$\begin{aligned}
\Delta f_2(t, x_2, (0,1)) =& \mu_2(s - \tilde{x}_1^F(t)) + \int_0^{\tilde{x}_1^F(t)+x_2(t)-s} \theta\left(u/s\right) du - \theta\left(\frac{\tilde{x}_1^N(t,(0,1)) + x_2(t) - s}{s}\right) x_2(t) \\
<& \mu_2(s - \tilde{x}_1^F(t)) + \theta\left(\tilde{x}_1^F(t) + x_2(t) - s\right)(\tilde{x}_1^F(t) + x_2(t) - s) - \theta\left(\frac{\tilde{x}_1^N(t,(0,1)) + x_2(t) - s}{s}\right) x_2(t) \\
<& \left(\mu_2 - \theta\left(\tilde{x}_1^F(t) + x_2(t) - s\right)\right)(s - \tilde{x}_1^F(t)) \leq 0.
\end{aligned}$$

iii. When $\tilde{x}_1^F(t) > s$, then

$$\Delta f_2(t,x_2,(0,1)) = \int_{\tilde{x}_1^F(t)-s}^{\tilde{x}_1^F(t)+x_2(t)-s} \theta(u/s)\,du - \theta\left(\frac{\tilde{x}_1^N(t,(0,1))+x_2(t)-s}{s}\right)x_2(t)$$
$$< \int_{\tilde{x}_1^F(t)-s}^{\tilde{x}_1^F(t)+x_2(t)-s} \theta(u/s)\,du - \theta\left(\frac{\tilde{x}_1^F(t)+x_2(t)-s}{s}\right)x_2(t) < 0.$$

Thus, $\Delta f_2(t,x_2,(0,1)) \le 0$, i.e., $f_2^N(t,x_2,(0,1)) \le f_2^F(t,x_2)$, for all $t, x_2(t)$. Using a similar argument as we prove Lemma 4, we can find $t_5 \in [0,d)$ such that $\tilde{x}_2^N(t_5,(0,1)) < \tilde{x}_2^F(t_5)$ by contradiction via (61) with $k=2$. Apply Lemma 2 with $y(t)=\tilde{x}_2^F(t), z(t)=\tilde{x}_2^N(t,(0,1)),\ f(t,y)=f_2^F(t,x_2)$, and $g(t,z)=f_2^N(t,x_2,(0,1))$, we obtain the desired result.

2. When $\beta_1 \ge 0.5$, for HP customers, we prove the results by Lemmas 1 and 2. To apply these Lemmas, we need to rewrite the HP net flow rate functions f_1^I as a function of $x_1(t)$ (instead of the two-dimensional variable $x(t)$). By (7) and (9) we know that, $\tilde{x}_1^F(t)$ (so as $f_1^F(t,x)$) is independent of $\tilde{x}_2^F(t)$, while $\tilde{x}_1^N(t,\boldsymbol{\beta})$ (so as $f_1^N(t,x,\boldsymbol{\beta})$) depends on $\tilde{x}_2^N(t,\boldsymbol{\beta})$. Therefore, we can directly rewrite f_1^F as a function of $x_1(t)$. For $I=N$, denote $\varphi_1(l,\boldsymbol{\beta})$ as the unique solution of (8) under N with $x_1(t)=l(t)$ as given for any non-negative periodic function $l(t)$ with period d. Then, we can rewrite f_1^N as a function of $x_1(t)$ with $x_2(t,\boldsymbol{\beta})=\varphi_1(x_1,\boldsymbol{\beta}) \ge 0$. Specifically, f_1^I can be rewritten as follows:

$$f_1^N(t,x_1,\boldsymbol{\beta}) = \lambda_1(t) - \mu_1(x_1(t)\wedge s) - \theta\left(\beta_1\frac{(x_1(t)+\varphi_1(x_1(t),\boldsymbol{\beta})-s)^+}{s}\right)(x_1(t)-s)^+,$$
$$f_1^F(t,x_1) = \lambda_1(t) - \mu_1(x_1(t)\wedge s) - \int_0^{(x_1(t)-s)^+} \theta(u/s)\,du.$$

Since θ is increasing and concave, $\varphi_1(x_1,\boldsymbol{\beta}) \ge 0$, it is obvious that $f_1^N(t,\boldsymbol{\beta}) \le f_1^F(t,x_1)$ for any $t \ge 0$ when $\beta_1 \ge 0.5$. Using a similar argument as we prove case 3 of Lemma 4 by applying Lemmas 1 and 2, we can obtain that $\tilde{x}_1^N(t,\boldsymbol{\beta}) \le \tilde{x}_1^F(t)\,\forall t$, and $\bar{x}_1^N(\boldsymbol{\beta}) < \bar{x}_1^F$ if $\max_{t\ge 0}\tilde{x}_1^F(t) > s$.

For LP customers: similar to how we handle the HP class, we can rewrite $f_2^N(t,x,\boldsymbol{\beta})$ as a function of $x_2(t)$ with $x_1(t)=\varphi_2(x_2(t))$, where $\varphi_2(l(t))$ denote the unique solution of (7) under N with $x_2(t)=l(t)$ as given, for any non-negative periodic function $l(t)$. Specifically,

$$f_2^N(t,x_2,\boldsymbol{\beta}) = \lambda_2(t) - \mu_2\left((s-\varphi_2(x_2(t)))^+ \wedge x_2(t)\right) - \theta\left(\beta_2\frac{(\varphi_2(x_2(t))+x_2(t)-s)^+}{s}\right)(x_2(t)-(s-\varphi_2(x_2(t)))^+)^+.$$

We can also rewrite $f_2^F(t,x)$ as a function of $x_2(t)$ with $x_1(t)=\tilde{x}_1^F(t)$ and define $\Delta f_2(t,x_2,\boldsymbol{\beta}) := f_2^N(t,x_2,\boldsymbol{\beta}) - f_2^F(t,x_2)$ like we did in part 1 of this proof.

(a) When $\beta_2 = 0$, if $\max_{t\ge 0}\tilde{x}_1^F(t) > s$ and $\mu_2 \le \theta(0)$, then $\tilde{x}_2^F(t) < \tilde{x}_2^N(t,\boldsymbol{\beta})\,\forall t$. Note that, Lemma 9 implies that $\varphi_2(x_2(t)) \le \varphi_2(0) < \tilde{x}_1^F(t), \forall t,\ x_2(t) > 0$. Using this set of inequalities and a similar analysis as Case 1.(b), we can obtain the desired results by Lemma 2 with $y(t)=\tilde{x}_2^N(t,(\beta_1,0)), z(t)=\tilde{x}_2^F(t),\ f(t,y)=f_2^N(t,x_2,(\beta_1,0))$, and $g(t,z)=f_2^F(t,x_2)$.

(b) When $\beta_2 = 1$,

i. If $\min\limits_{t\ge 0}\tilde{x}_1^N(t,\boldsymbol{\beta}) \ge s$, we show that there exists a threshold $\tilde{\rho}_2^1$ such that if $\bar{\rho}_2 < \tilde{\rho}_2^1$, then $\tilde{x}_2^F(t) < \tilde{x}_2^N(t,\boldsymbol{\beta}), \forall t$. Let $\delta_1(t) := \tilde{x}_1^F(t) - \varphi_2(t,\boldsymbol{\beta},0)$ and $\underline{\delta}_1 := \min_t \delta_1(t)$. Since $\varphi_2(x_2(t)) < \varphi_2(0) < \tilde{x}_1^F(t), \forall t$, we have $\tilde{x}_1^F(t) > \varphi_2(x_2(t)) + \underline{\delta}_1, \forall t$. We prove this result by three steps: (1) When $\tilde{x}_2^F(t) < \underline{\delta}_1$, for $t \ge 0$, we show that $\tilde{x}_2^F(t) < \tilde{x}_2^N(t,(\beta_1,1))$ by Lemma 2. (2) For the special case when the LP arrival rate is stationary with $\rho_2(t) = \bar{\rho}_2$, for $t \geq 0$, denote the LP periodic equilibrium of the corresponding system as $\hat{x}_2^F(t)$. Then, we show that there exists a threshold $\tilde{\rho}_2^1$ for the LP load such that $\hat{x}_2^F(t) < \underline{\delta}_1$ if $\bar{\rho}_2 < \tilde{\rho}_2^1$. (3) When the LP arrival rate is non-stationary with maximum LP load $\bar{\rho}_2$, we show that $\tilde{x}_2^F(t) \leq \hat{x}_2^F(t)$ by Lemma 1. Therefore, if $\bar{\rho}_2 < \tilde{\rho}_2^1$, then $\tilde{x}_2^F(t) < \underline{\delta}_1$, which further implies the desired result.

Step 1: Let $y(t) = \tilde{x}_2^N(t,(\beta_1,1))$, $z(t) = \tilde{x}_2^F(t)$, $f(t,z) = f_2^N(t,\tilde{x}_2^F(t),(\beta_1,1))$, and $g(t,z) = f_2^F(t,\tilde{x}_2^F(t))$. To apply Lemma 2, we need to show as follows that $f_2^F(t,\tilde{x}_2^F(t)) \leq f_2^N(t,\tilde{x}_2^F(t),(\beta_1,1))$, for $t \geq 0$, and there exists t_6 such that $\tilde{x}_2^F(t_6) < \tilde{x}_2^N(t_6,(\beta_1,1))$.

When $\min_{t\geq 0} \tilde{x}_1^F(t) > \min_{t\geq 0} \tilde{x}_1^N(t,\boldsymbol{\beta}) \geq s$, $\beta_2 = 1$, and $\tilde{x}_2^F(t) < \underline{\delta}_1 < \tilde{x}_1^F(t) - \varphi_2(\tilde{x}_2^F(t))$, we have

$$\begin{aligned}\Delta f_2(t,\tilde{x}_2^F(t),(\beta_1,1)) =& \int_{\tilde{x}_1^F(t)-s}^{\tilde{x}_1^F(t)+\tilde{x}_2^F(t)-s} \theta\left(u/s\right) du - \theta\left(\frac{\varphi_2(\tilde{x}_2^F(t)) + \tilde{x}_2^F(t) - s}{s}\right)\tilde{x}_2^F(t)(t)\\ >&\theta\left(\frac{\tilde{x}_1^F(t)-s}{s}\right)\tilde{x}_2^F(t) - \theta\left(\frac{\tilde{x}_1^F(t)-s}{s}\right)\tilde{x}_2^F(t) = 0.\end{aligned}$$

Next, we show by contradiction the existence of t_6. If on the contrary, $\tilde{x}_2^F(t) \geq \tilde{x}_2^N(t,(\beta_1,1))$, for all t, then $\varphi_2(\tilde{x}_2^N(t,(\beta_1,1))) + \tilde{x}_2^N(t,(\beta_1,1)) < \varphi_2(t,(\beta_1,1),0) + \tilde{x}_2^F(t) \leq \varphi_2(t,(\beta_1,1),0) + \underline{\delta}_1 \leq \tilde{x}_1^F(t)$, and

$$\begin{aligned}f_2^F(t,\tilde{x}_2^F) - f_2^N(t,\tilde{x}_2^N,(\beta_1,1)) =& \theta\left(\frac{(\varphi_2(\tilde{x}_2^N(t,(\beta_1,1))) + \tilde{x}_2^N(t,(\beta_1,1)) - s)}{s}\right)\tilde{x}_2^N(t,(\beta_1,1)) - \int_{(\tilde{x}_1^F(t)-s)}^{(\tilde{x}_1^F(t)+\tilde{x}_2^F(t)-s)} \theta\left(u/s\right) du\\ \leq& \theta\left(\frac{(\tilde{x}_1^F(t)-s)}{s}\right)\tilde{x}_2^F(t) - \int_{(\tilde{x}_1^F(t)-s)}^{(\tilde{x}_1^F(t)+\tilde{x}_2^F(t)-s)} \theta\left(u/s\right) du < 0,\end{aligned}$$

which contradicts to (61).

Step 2: Since $\bar{x}_1^F(t) > s$, when the LP arrival rate is stationary, by (8), $\hat{x}_2^F(t)$ is the unique periodic solution of

$$\dot{x}_2^F = \bar{\rho}_2\mu_2 s - \int_{(\tilde{x}_1^F(t)-s)}^{(\tilde{x}_1^F(t)+x_2^F(t)-s)} \theta\left(u/s\right) du := \hat{f}_2^F(t,x_2^F(t)).$$

By the nature of periodic equilibrium we have

$$\bar{\rho}_2\mu_2 s = \frac{1}{d}\int_0^d \int_{(\tilde{x}_1^F(t)-s)}^{(\tilde{x}_1^F(t)+\hat{x}_2^F(t)-s)} \theta\left(u/s\right) du dt. \tag{62}$$

Thus, when $\bar{\rho}_2 \to 0$, we must have $\hat{x}_2^F(t) \to 0$ by continuity. Therefore, for any $\underline{\delta}_1 > 0$, there exists $\tilde{\rho}_2^1$ such that if $\bar{\rho}_2 < \tilde{\rho}_2^1$, we have $\hat{x}_2^F(t) < \underline{\delta}_1$ for $t \geq 0$.

Step 3: When the LP arrival rate is non-stationary with maximum LP load $\bar{\rho}_2$, we prove $\tilde{x}_2^F(t) \leq \hat{x}_2^F(t)$ by Lemma 1 with $y(t) = \tilde{x}_2^F(t)$, $z(t) = \hat{x}_2^F(t)$, $f(t,y) = f_2^F(t,y)$, and $g(t,z) = \hat{f}_2^F(t,z)$. It is clear that $f_2^F(t,y) \leq \hat{f}_2^F(t,z)$ since $f_2^F(t,x_2) - \hat{f}_2^F(t,x_2) = \lambda_2(t) - \bar{\rho}_2\mu_2 s \leq 0$. And we can show the existence of $t_{11} \in [0,d)$ such that $\tilde{x}_2^F(t_{11}) \leq \hat{x}_2^F(t_{11})$ by contradiction using (62) and $\int_0^d f_2^F(t,\tilde{x}_2^F(t)) = 0$ as follows:

$$\frac{1}{d}\int_0^d \int_{(\tilde{x}_1^F(t)-s)}^{(\tilde{x}_1^F(t)+\tilde{x}_2^F(t)-s)} \theta\left(u/s\right) du dt = \frac{1}{d}\int_0^d \lambda_2(t) dt < \bar{\rho}_2\mu_2 s = \frac{1}{d}\int_0^d \int_{(\tilde{x}_1^F(t)-s)}^{(\tilde{x}_1^F(t)+\hat{x}_2^F(t)-s)} \theta\left(u/s\right) du dt.$$

ii. If $\min_{t\geq 0} \tilde{x}_1^N(t,\boldsymbol{\beta}) \geq s$, or $\max_{t\geq 0} \tilde{x}_1^N(t,\boldsymbol{\beta}) > s$ and $\mu_2 \geq \theta(\infty)$, there exists a threshold $\tilde{\rho}_2^2$ such that if $\underline{\rho}_2 > \tilde{\rho}_2^2$, then $\tilde{x}_2^F(t) > \tilde{x}_2^N(t,\boldsymbol{\beta}), \forall t$. By Lemma 9, when $\max_{t\geq 0} \tilde{x}_1^N(t,\boldsymbol{\beta}) > s$, $\varphi_2(\infty) < \varphi_2(x_2(t)) < \tilde{x}_1^F(t)$, for $x_2(t) > 0$. Let $\delta_2(t) := \tilde{x}_1^F(t) - \varphi_2(\infty) > 0$ and $\bar{\delta}_2 := \max_t \delta_2(t)$. We prove this result by three steps: (1)

When $\tilde{x}_2^N(t,(\beta_1,1)) > 2\bar{\delta}_2$, we show that $\tilde{x}_2^F(t) > \tilde{x}_2^N(t,(\beta_1,1))$ using Lemma 2. (2) For the special case when the LP arrival rate is stationary with $\rho_2(t) = \underline{\rho}_2$, for $t \geq 0$, denote the LP periodic equilibrium of the corresponding system as $\breve{x}_2^F(t)$. Then, we show that there exists a threshold $\tilde{\rho}_2^2$ for the LP load such that $\breve{x}_2^N(t,(\beta_1,1)) > 2\bar{\delta}_2$ if $\underline{\rho}_2 > \tilde{\rho}_2^2$. (3) When the LP arrival rate is non-stationary with minimum LP load $\underline{\rho}_2$, we show that $\tilde{x}_2^N(t,(\beta_1,1)) \geq \breve{x}_2^N(t,(\beta_1,1))$. Therefore, if $\underline{\rho}_2 > \tilde{\rho}_2^2$, then $\tilde{x}_2^N(t,(\beta_1,1)) > 2\bar{\delta}_2$, which further implies the desired result. The proofs of Steps 2 and 3 are similar to our proofs of Steps 2 and 3 for Case 2.(b).i, thus we omit it here for brevity.

Step 1: Let $y(t) = \tilde{x}_2^F(t)$, $z(t) = \tilde{x}_2^N(t,(\beta_1,1))$, $f(t,z) = f_2^F(t,\tilde{x}_2^N(t,(\beta_1,1)))$, and $g(t,z) = f_2^N(t,\tilde{x}_2^N(t,(\beta_1,1)))$. To apply Lemma 2, we need to show that when $\tilde{x}_2^N(t,(\beta_1,1)) \geq 2\bar{\delta}_2$, $\min_{t\geq 0} \tilde{x}_1^N(t,\boldsymbol{\beta}) \geq s$, or $\max_{t\geq 0} \tilde{x}_1^N(t,\boldsymbol{\beta}) > s$ and $\mu_2 \geq \theta(\infty)$, the following two conditions are satisfied: $f_2^F(t,\tilde{x}_2^N) \geq f_2^N(t,\tilde{x}_2^N,(\beta_1,1))$ and there exists $t_7 \in [0,d)$ such that $\tilde{x}_2^F(t_7) > \tilde{x}_2^N(t_7,(\beta_1,1))$. For $\tilde{x}_2^N(t,(\beta_1,1)) > 0$,

- When $\varphi_2(\tilde{x}_2^N(t,(\beta_1,1))) < \tilde{x}_1^F(t) \leq s$,

  —If $\tilde{x}_2^N(t,(\beta_1,1)) \leq s - \tilde{x}_1^F(t)$, then $\Delta f_2(t,\tilde{x}_2^N,(\beta_1,1)) = 0$ is trivial.

  —If $s - \tilde{x}_1^F(t) < \tilde{x}_2^N(t,(\beta_1,1)) \leq s - \varphi_2(\tilde{x}_2^N(t,(\beta_1,1)))$, then when $\mu_2 \geq \theta(\infty)$,

$$\Delta f_2(t,\tilde{x}_2^N,(\beta_1,1)) = \int_0^{(\tilde{x}_1^F(t)+\tilde{x}_2^N(t,(\beta_1,1))-s)} \theta\,(u/s)\,du - \mu_2(\tilde{x}_1^F(t) + \tilde{x}_2^N(t,(\beta_1,1)) - s) < 0.$$

  —If $\tilde{x}_2^N(t,(\beta_1,1)) > s - \varphi_2(\tilde{x}_2^N(t,(\beta_1,1)))$, then when $\mu_2 \geq \theta(\infty)$,

$$\begin{aligned}
\Delta f_2(t,\tilde{x}_2^N,(\beta_1,1)) =& \mu_2(\varphi_2(\tilde{x}_2^N(t,(\beta_1,1))) - \tilde{x}_1^F(t)) + \int_0^{(\tilde{x}_1^F(t)+\tilde{x}_2^N(t,(\beta_1,1))-s)} \theta\,(u/s)\,du \\
&- \theta\left(\frac{(\varphi_2(\tilde{x}_2^N(t,(\beta_1,1))) + \tilde{x}_2^N(t,(\beta_1,1)) - s)}{s}\right)(\varphi_2(\tilde{x}_2^N(t,(\beta_1,1))) + \tilde{x}_2^N(t,(\beta_1,1)) - s) \\
<& \mu_2(\varphi_2(\tilde{x}_2^N(t,(\beta_1,1))) - \tilde{x}_1^F(t)) \\
&+ \int_0^{(\tilde{x}_1^F(t)+\tilde{x}_2^N(t,(\beta_1,1))-s)} \theta\,(u/s)\,du - \int_0^{\varphi_2(\tilde{x}_2^N(t,(\beta_1,1)))+\tilde{x}_2^N(t,(\beta_1,1))-s} \theta\,(u/s)\,du \\
=& -\mu_2(\tilde{x}_1^F(t) - \varphi_2(\tilde{x}_2^N(t,(\beta_1,1)))) + \int_{\varphi_2(\tilde{x}_2^N(t,(\beta_1,1)))+\tilde{x}_2^N(t,(\beta_1,1))-s}^{(\tilde{x}_1^F(t)+\tilde{x}_2^N(t,(\beta_1,1))-s)} \theta\,(u/s)\,du \\
\leq& (\theta(\infty) - \mu_2)(\tilde{x}_1^F(t) - \varphi_2(\tilde{x}_2^N(t,(\beta_1,1)))) \leq 0.
\end{aligned}$$

- When $\varphi_2(\tilde{x}_2^N(t,(\beta_1,1))) \leq s < \tilde{x}_1^F(t)$,

  —If $\tilde{x}_2^N(t,(\beta_1,1)) \leq s - \varphi_2(\tilde{x}_2^N(t,(\beta_1,1)))$, then when $\mu_2 \geq \theta(\infty)$,

$$\Delta f_2(t,\tilde{x}_2^N,(\beta_1,1)) = \int_{(\tilde{x}_1^F(t)-s)}^{(\tilde{x}_1^F(t)+\tilde{x}_2^N(t,(\beta_1,1))-s)} \theta\,(u/s)\,du - \mu_2\tilde{x}_2^N(t,(\beta_1,1)) < (\theta(\infty) - \mu_2)\tilde{x}_2^N(t,(\beta_1,1)) \leq 0.$$

  —If $\tilde{x}_2^N(t,(\beta_1,1)) > s - \varphi_2(\tilde{x}_2^N(t,(\beta_1,1)))$, then when $\mu_2 \geq \theta(\infty)$ and $\tilde{x}_2^N(t,(\beta_1,1)) \geq 2\bar{\delta}_2$,

$$\begin{aligned}
\Delta f_2(t,\tilde{x}_2^N,(\beta_1,1)) =& -\mu_2(s - \varphi_2(\tilde{x}_2^N(t,(\beta_1,1)))) + \int_{(\tilde{x}_1^F(t)-s)}^{(\tilde{x}_1^F(t)+\tilde{x}_2^N(t,(\beta_1,1))-s)} \theta\,(u/s)\,du \\
&- \theta\left(\frac{(\varphi_2(\tilde{x}_2^N(t,(\beta_1,1))) + \tilde{x}_2^N(t,(\beta_1,1)) - s)}{s}\right)(\varphi_2(\tilde{x}_2^N(t,(\beta_1,1))) + \tilde{x}_2^N(t,(\beta_1,1)) - s) \\
<& (\theta(\infty) - \mu_2)(s - \varphi_2(\tilde{x}_2^N(t,(\beta_1,1)))) \\
&+ \int_{(\tilde{x}_1^F(t)-s)}^{(\tilde{x}_1^F(t)+\tilde{x}_2^N(t,(\beta_1,1))-s)} \theta\,(u/s)\,du - \theta\left(\frac{(\varphi_2(\tilde{x}_2^N(t,(\beta_1,1))) + \tilde{x}_2^N(t,(\beta_1,1)) - s)}{s}\right)\tilde{x}_2^N(t,(\beta_1,1)) \\
<& \left(\theta\left(\frac{(\tilde{x}_1^F(t) + \frac{1}{2}\tilde{x}_2^N(t,(\beta_1,1)) - s)}{s}\right) - \theta\left(\frac{(\varphi_2(\tilde{x}_2^N(t,(\beta_1,1))) + \tilde{x}_2^N(t,(\beta_1,1)) - s)}{s}\right)\right)\tilde{x}_2^N(t,(\beta_1,1)) \leq 0.
\end{aligned}$$

Note that $\tilde{x}_2^N(t,(\beta_1,1)) \geq 2\bar{\delta}_2 > 2(\tilde{x}_1^F(t) - \varphi_2(\tilde{x}_2^N(t,(\beta_1,1))))$ implies the last inequality.

Table 4 **Comparison of the simulation- and fluid-based average system abandonment rankings ($\mu_1 = \mu_2 = 1$, $s = 20$, $\theta(x) = 5 - 4e^{-x}$, and $\lambda_k(t) = \rho_k \mu_k s(1 - 0.8\sin(\pi t/12))$).**

ρ_1	ρ_2	Class (k)	simulation-based ranking		fluid-based ranking	
			$\bar{\mathcal{A}}_k^N$	$\bar{\mathcal{A}}_k^F$	$\bar{\mathcal{A}}_k^N$	$\bar{\mathcal{A}}_k^F$
1.2	0.1	1	8.3821	8.3358	8.1141	8.0954
1.2	0.8	1	8.4313	8.3415	8.1318	8.0954
1.2	1.2	1	8.4391	8.3413	8.1354	8.0745
1.2	0.1	2	1.6026	1.6186	1.6409	1.6471
1.2	0.8	2	13.3838	13.4917	13.9541	13.9748
1.2	1.2	2	20.4645	20.6164	21.3479	21.9546

- When $\varphi_2(\tilde{x}_2^N(t,(\beta_1,1))) > s$, since $\tilde{x}_2^N(t,(\beta_1,1)) \geq 2\bar{\delta}_2$,

$$\Delta f_2(t,\tilde{x}_2^N,(\beta_1,1)) = \int_{(\tilde{x}_1^F(t)-s)}^{(\tilde{x}_1^F(t)+\tilde{x}_2^N(t,(\beta_1,1))-s)} \theta(u/s)\,du - \theta\left(\frac{(\varphi_2(\tilde{x}_2^N(t,(\beta_1,1))) + \tilde{x}_2^N(t,(\beta_1,1)) - s)}{s}\right)\tilde{x}_2^N(t,(\beta_1,1))$$
$$< \left(\theta\left(\frac{(\tilde{x}_1^F(t) + \frac{1}{2}\tilde{x}_2^N(t,(\beta_1,1)) - s)}{s}\right) - \theta\left(\frac{(\varphi_2(\tilde{x}_2^N(t,(\beta_1,1))) + \tilde{x}_2^N(t,(\beta_1,1)) - s)}{s}\right)\right)\tilde{x}_2^N(t,(\beta_1,1)) \leq 0.$$

To summarize, we show that $\Delta f_2(t,\tilde{x}_2^N,(\beta_1,1)) < 0$ when $\tilde{x}_2^N(t,(\beta_1,1)) \geq 2\bar{\delta}_2$, $\min_{t\geq 0}\tilde{x}_1^N(t,\boldsymbol{\beta}) \geq s$, or $\max_{t\geq 0}\tilde{x}_1^N(t,\boldsymbol{\beta}) > s$ and $\mu_2 \geq \theta(\infty)$.

The existence of t_7 can be shown in a similar analysis as we prove Step 1 of Case 2.(b).i, therefore we omit it here for brevity. $\square$

Appendix D: Supplementary Numerical Results for Section 7.2

In this section, we provide supplementary numerical results for the robustness check of our fluid approximations for finite stochastic systems.

In Section 7.2, we have verified the robustness of our fluid-based number-in-system ranking results for moderately-sized systems, i.e., $s = 20$. Here, we consider numerical examples with alternative system sizes, i.e., $s = 10, 50$.

In Figures 12 and 13, we plot 95% confidence intervals for the expected number-in-system process, under each information level, along with their corresponding time-dependent fluid limits, at equilibrium, over one period, for small ($s = 10$) and large ($s = 50$) server sizes. Other parameters are consistent with our numerical examples in Figures 10 and 11.

As can be seen in Figures 12 and 13, the fluid approximations are effective to describe performance in a stochastic system even with small-sized system ($s = 10$), and is pretty precise with large-sized system ($s = 50$). And the rankings of the simulated average number-in-system stochastic processes are consistent with the fluid-based ranking, at each point in time, for both system sizes.

As for the robustness of the fluid-based average system abandonment rate rankings for the stochastic systems, we estimate the expected average system abandonment rates of the stochastic systems for each information level and obtain their rankings (statistically significant at 95% confidence level) using simulation. The results are shown in Table 4.

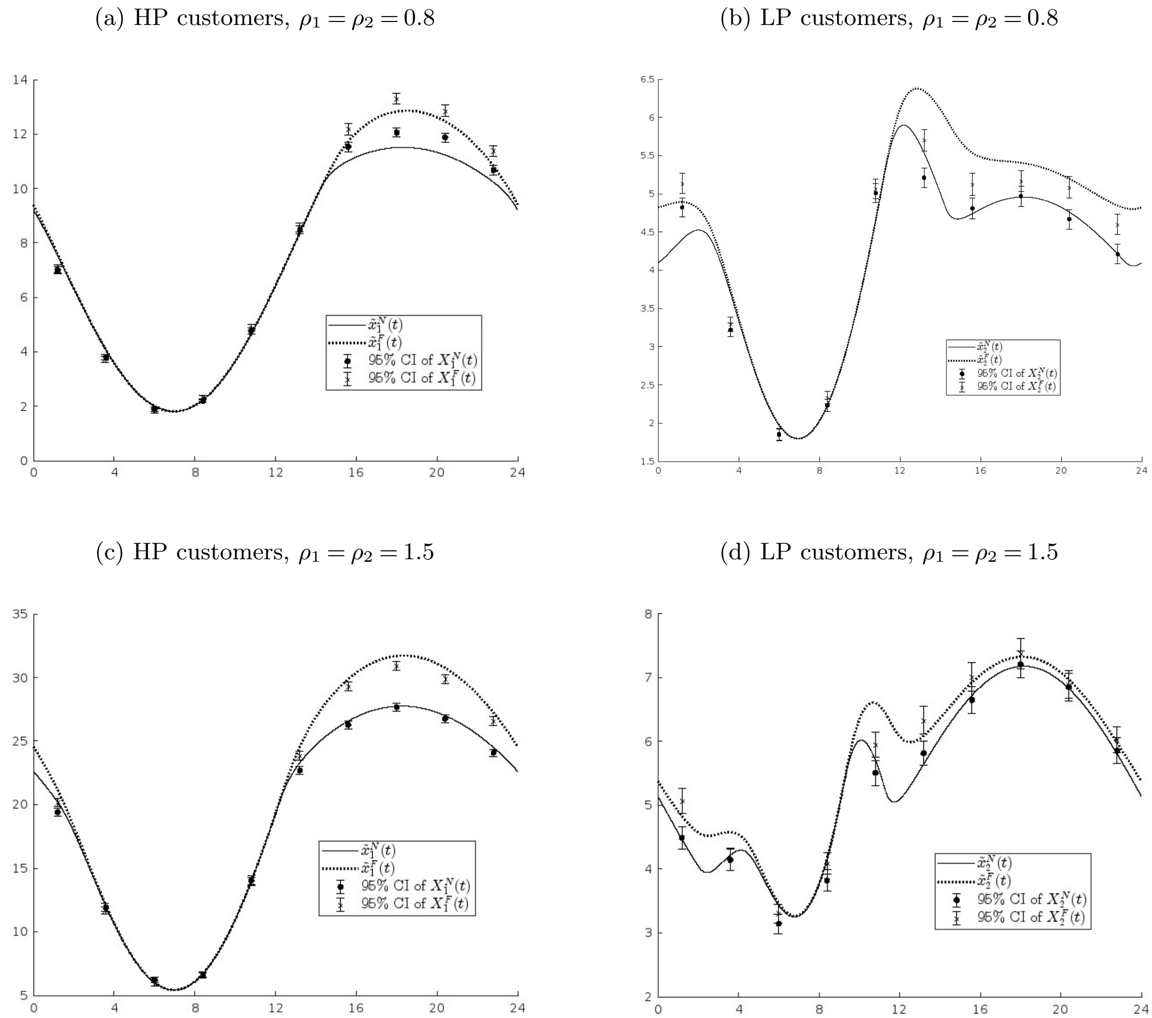

Figure 12 Comparisons of the number-in-system trajectories under different information levels for small size simulated stochastic systems and the fluid models ($s = 10$**,** $\mu_1 = \mu_2 = 1$**,** $\theta(x) = 5 - 4e^{-x}, \lambda_k(t) = \rho_k \mu_k s(1 - 0.8 \sin(\pi t/12))$**).**

Table 4 shows that full information yields lower HP average system abandonment and higher LP average system abandonment than no information in both simulation- and fluid-based systems, across all three cases where LP load ranges from low to high. In conclusion, Table 4 indicates that our fluid-based average system abandonment rankings are valid for small-sized stochastic systems.

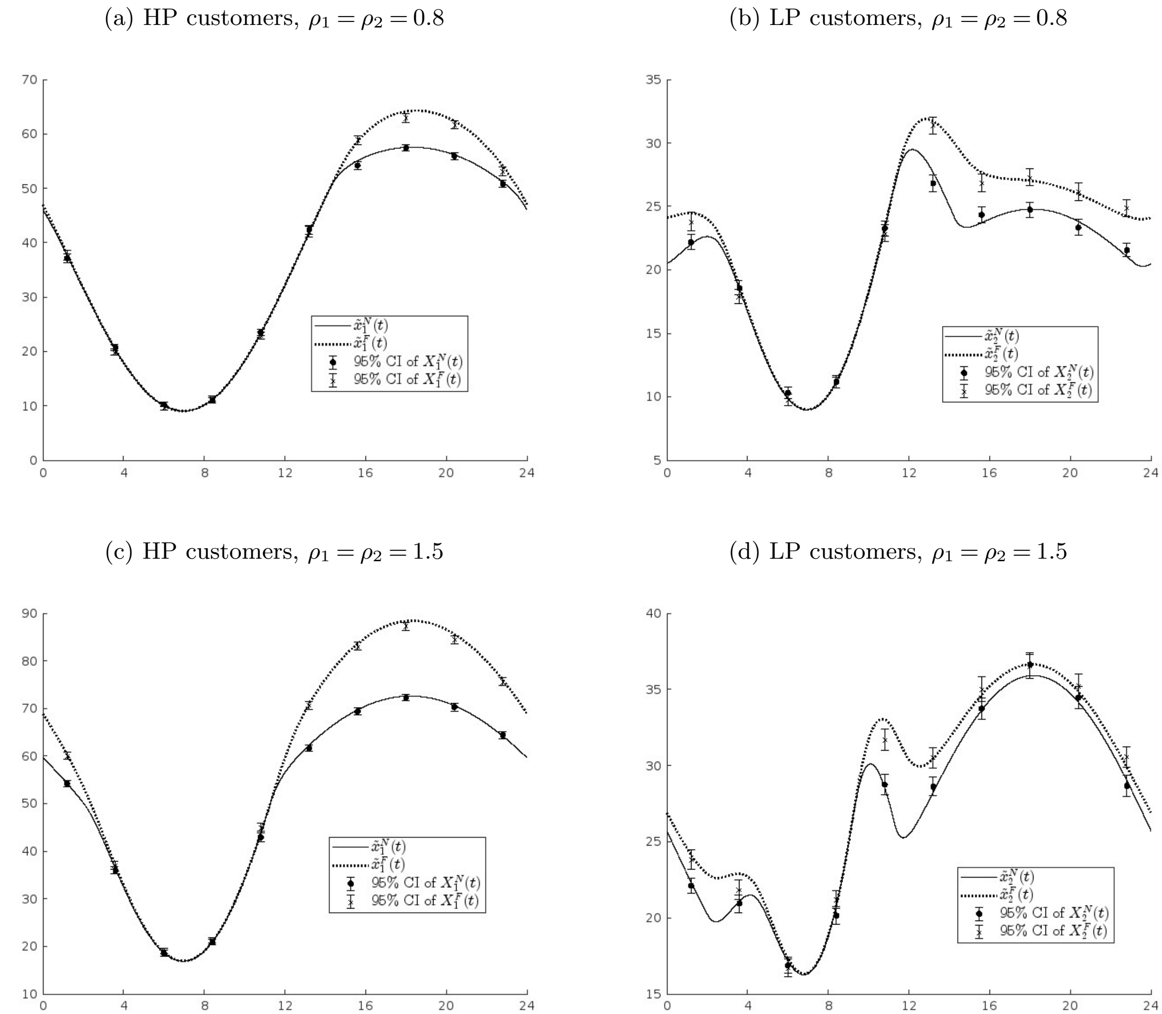

Figure 13 Comparisons of the number-in-system trajectories under different information designs for large size simulated stochastic systems and the fluid models ($s = 50$, $\mu_1 = \mu_2 = 1$, $\theta(x) = 5 - 4e^{-x}, \lambda_k(t) = \rho_k \mu_k s(1 - 0.8 \sin(\pi t/12))$**).**