Does the Past Predict the Future? The Case of Delay Announcements in Service Systems
SUPPLEMENTARY MATERIAL

Abstract from main paper

Motivated by the recent interest in making delay announcements in large service systems, such as call centers, we investigate the accuracy of announcing the waiting time of the Last customer to Enter Service (LES). In practice, customers typically respond to delay announcements by either balking or by becoming more or less impatient, and their response alters system performance. We study the accuracy of the LES announcement in single-class multi-server Markovian queueing models with announcement-dependent customer behavior. We show that, interestingly, even in this stylized setting, the LES announcement *may not* always be accurate. This motivates the need to study its performance carefully, and to determine conditions under which it is accurate. Since the direct analysis of the system with customer response is prohibitively difficult, we focus on many-server heavy-traffic analysis instead. We consider the quality-and-efficiency-driven (QED) and the efficiency-driven (ED) many-server heavy-traffic regimes and prove, under both regimes, that the LES prediction is asymptotically accurate if, and only if, asymptotic fluctuations in the queue length process are small. This result provides an easy check for the accuracy of LES in practice. We go further by establishing initial conditions which guarantee the asymptotic accuracy of LES. We supplement our theoretical results with an extensive simulation study.

1. Outline of this Supplement

In §2, we consider the system without customer response and develop an adjustment to the LES announcement which performs better than LES when the system is initialized away from its steady state, at fluid scale. In §3, we present tables corresponding to Figures 2 and 3 from the main paper. Finally, in §4, we present additional simulation results which relate the queue-length and wait-time relative errors for alternative model parameters such as the traffic intensity, the number of servers, and time-varying characteristics of the arrival process.

2. Adjustment to the LES Announcement in the ED Regime without Customer Response

When the system has not yet reached its steady state, at fluid scale, we can exploit the dynamics of the $M/M/N+M$ fluid model to develop adjustments for the LES prediction. In §2.2, we show that those adjustments are more accurate than the LES prediction.

Let $w(s)$ denote the waiting time, at time s, in the fluid model. We use the same notation as above, but omit dependence on N. To simplify notation, we do a time change so that $\tau_t \equiv 0$. We proceed as follows.

Step 1. Express $q_{LES} \equiv q(\tau_t)$ in terms of $w_{LES} \equiv w(\tau_t)$.

Let $q_o(s)$ denote the queue length in the fluid model at time $s \geq 0$ with the arrival process turned off at $\tau_t \equiv 0$. Then, $q_o(s)$ must satisfy the ODE in (2.1) with initial condition $q_o(0) = q_{LES}$:

$$q'(s) \equiv \frac{dq(s)}{ds} = \rho - 1 - \theta q(s) \ , \tag{2.1}$$

The ODE in (2.1) has the solution

$$q_o(s) = (q_{LES} + \frac{1}{\theta})e^{-\theta s} - \frac{1}{\theta} \ , \ \text{for } s \geq 0 \ . \tag{2.2}$$

Noting that w_{LES} is the first passage time to the zero state, we get that

$$q_o(w_{LES}) = 0 \ . \tag{2.3}$$

Plugging in w_{LES} and solving (2.3) for q_{LES} yields that

$$q_{LES} = \frac{1}{\theta}e^{\theta w_{LES}} - \frac{1}{\theta} \ . \tag{2.4}$$

Step 2. Express $q(t) = q(w_{LES})$ in terms of w_{LES}.

We assume here that the new customer arrives exactly when the LES customer enters service. This assumption is reasonable because the difference between the arrival times of the LES and current customer is asymptotically negligible in the ED limit. We know that $q(t)$ must satisfy the ODE in (2.1) with initial queue length equal to q_{LES}. We solve (2.1) and substitute in w_{LES}:

$$q(w_{LES}) = \frac{\rho - 1}{\theta} + (q_{LES} - \frac{\rho - 1}{\theta})e^{-\theta w_{LES}} \ . \tag{2.5}$$

Plugging in the value of q_{LES} in (2.4) yields:

$$q(t) = q(w_{LES}) = \frac{\rho}{\theta} - \frac{\rho}{\theta}e^{-\theta w_{LES}} \ . \tag{2.6}$$

Step 3. Express $w(t)$ in terms of w_{LES}.

There remains to solve for $w(t)$ in terms of w_{LES}. Let $q_o^t(s)$ denote the queue length in the fluid model at time $t+s$, $s \geq 0$, with the arrival process turned off at t. Then, $q_o^t(s)$ must satisfy the ODE in (2.7) with initial condition $q_o^t(0) = q(t) = q(w_{LES})$ given in (2.6).

$$\frac{dq(u)}{du} = -1 - \theta q(u) \ , \tag{2.7}$$

The solution to that ODE is given by

$$q_o^t(s) = (q(t) + \frac{1}{\theta})e^{-\theta s} - \frac{1}{\theta} \text{ for } s \geq 0 \ . \tag{2.8}$$

But, $w(t)$ is the first passage time from state $q_o^t(0)$ to state 0, so $w(t)$ must solve the equation

$$q_o^t(w(t)) = 0 \ , \tag{2.9}$$

for $q_o^t(s)$ in (2.8). Plugging in the value of $q(t)$ in (2.6) and solving for $w(t)$ yields:

$$w(t) = \frac{1}{\theta}\ln(\rho + 1 - \rho e^{-\theta w_{LES}}) \ . \tag{2.10}$$

For a given LES announcement w, our first adjusted announcement, LES$_a$, is given by:

$$\text{LES}_a \text{ announcement} = \frac{1}{\theta}\ln(\rho + 1 - \rho e^{-\theta w}) \ . \tag{2.11}$$

If $w = \ln(\rho)/\theta$ (steady-state fluid limit), then the LES announcement and its adjustment coincide. It is important to note that letting ρ approach 1 in (2.11) yields the adjustment for LES in the QED regime.

2.1. Alternative Adjustments

The improved performance of the adjusted LES announcement of §2 comes at the expense of exploiting more information about the system, such as the abandonment rate and the traffic intensity; see (2.11). To overcome this tradeoff between efficiency and ease of implementation, we experimented with several adjustments which do not rely on this additional information.

We considered announcements based on several past LES delays (typically two or three) experienced by successive customers in the system. We fit linear, quadratic, and exponential functions to those delays (as a function of the time of arrival to the system), and extrapolated those functions to the time of arrival of the current customer. We did so to obtain adjusted announcements based on additional past delays besides the most recent LES delay. Here, we do not include a separate discussion for those adjusted announcements because numerous simulation experiments indicated that they tend to be more accurate than LES in the transient state of the system, but less accurate than LES in steady state. In practice, it is not clear how to determine whether or not a system is in its steady state. Thus, those adjustments may not be very useful in a real-life system.

2.2. LES versus Adjustments

In Tables 1 and 2, we present point and 95% confidence interval estimates for the ASE of the LES and LES_a predictors. Our estimates are based on 10 independent simulation replications. We focus on both the transient and steady states of the $M/M/500+M$ model: We increase the number of arrivals in our sample, from 2000 to 50000. With a large number of arrivals, the system approaches its steady state. We compare ASE(LES) and ASE(LES_a) for each sample. We consider both the QED (Table 1) and ED (Table 2) limiting regimes.

We consider three different initial conditions for our systems: (i) the system is initially empty; (ii) all servers are initially busy but the queue is initially empty; (iii) all servers are initially busy, and the initial number of customers in queue is equal to the number of servers (in our system, 500). We consider a relatively large number of servers because fluid approximations are relatively accurate in that case. We expect to have ASE(LES) $\approx$ ASE(LES_a) when the system approaches its steady state. In that case, LES is asymptotically accurate. We let the traffic intensity $\rho = 1.0$ for the QED regime, and $\rho = 1.4$ for the ED regime.

Table 1 shows that LES_a is more effective than LES when the state of the system is far from its steady state condition; e.g, in the QED regime, this happens with busy servers and a long queue. Table 1 shows that, with all servers initially busy and a long initial queue, ASE(LES)/ASE(LES_a) decreases from roughly 11 to roughly 3 as the number of arrivals in our sample ranges from 2000 to 50000. That is, as expected, as the system approaches its steady state condition, the difference in performance between LES and LES_a decreases.

Consistent with Table 1, Table 2 shows that, in the ED regime, LES_a is generally more effective than LES. In particular, the difference in performance between LES and LES_a is most significant when the system is farthest from its steady state condition. Table 2 shows that, with an initially empty system, ASE(LES)/ASE(LES_a) ranges from roughly 3 to 1.2 as the number of arrivals in our sample ranges from 2000 to 50000. The difference in performance between LES and LES_a is smaller when the system is initialized in its steady state condition: ASE(LES)/ASE(LES_a) ranges from roughly 2 to 1.2 when all servers are initially busy and the queue length is initially large.

	Empty system		*N servers busy, QL = 0*		*N servers busy, QL = N*	
n	ASE(LES)	ASE(LES$_a$)	ASE(LES)	ASE(LES$_a$)	ASE(LES)	ASE(LES$_a$)
2000	0.787	0.792	1.33	1.32	102	9.23
	±0.40	±0.41	±0.48	±0.47	± 19	± 2.5
5000	2.38	2.51	1.83	1.81	45.4	5.68
	±0.85	± 0.91	±0.33	± 0.34	± 9.9	± 1.3
10000	2.32	2.38	1.81	1.82	24.6	3.92
	±0.63	±0.68	±0.21	±0.22	± 3.7	± 0.71
50000	2.16	2.18	1.99	2.02	7.36	2.64
	± 0.16	±0.17	± 0.15	± 0.16	± 0.93	± 0.14

Table 1: Comparison of the LES and LES$_a$ predictors in the $M/M/500+M$ model in the QED limiting regime ($\rho = 1.0$) with no customer reactions. The ASE's are reported in units of 10^{-4}. We vary the number of arrivals, n, in our sample and the initial state of the system.

3. Supporting Tables for Figures 2 and 3 of the main paper

In Tables 3 and 4, we present point and 95% confidence interval estimates corresponding to Figures 2 and 3 of the main paper. We also include corresponding values for RASE(LES). Tables 3 and 4 both show that RASE(LES) decreases as N increases. This illustrates the asymptotic accuracy of the LES announcement, and substantiates our theoretical results from the previous sections. Table 3 shows that, in the QED regime, the convergence of the RASE to 0 is relatively slow: This is because the magnitude of waiting times is asymptotically small in the QED limit.

4. Simulation Study Relating Queue-Length and Wait-Time Errors

In this section (Tables 5-10), we conduct a detailed simulation study which aims at gaining more concrete insights into the respective magnitudes of the queue-length and wait-time relative errors corresponding to our main asymptotic result from the main paper. The objective behind this simulation study is to assess how the relative error in the queue length translates into the accuracy of the LES prediction. Based on our simulation study, we can now answer questions such as: Under given modelling assumptions, if the reported queue-length error is less than, say, 5%, what can we say about the corresponding relative wait-time error? We provide some additional explanation below.

	Empty system		*N servers busy, QL = 0*		*N servers busy, QL = N*	
n	ASE(LES)	ASE(LES$_a$)	ASE(LES)	ASE(LES$_a$)	ASE(LES)	ASE(LES$_a$)
2000	2.00	0.652	3.79	0.781	3.39	1.79
	± 0.74	±0.14	±0.66	±0.31	± 1.5	±0.6
5000	2.90	1.07	3.19	1.45	2.13	1.46
	± 0.45	±0.18	± 0.88	±0.63	±0.50	± 0.40
10000	2.46	1.48	2.70	1.70	2.15	1.72
	±0.45	±0.38	±0.65	±0.52	± 0.46	± 0.39
50000	2.36	1.92	2.34	1.89	2.37	2.00
	± 0.15	±0.15	±0.21	±0.18	±0.25	± 0.21

Table 2: Comparison of the LES and LES$_a$ predictors in the $M/M/500 + M$ model in the ED limiting regime ($\rho = 1.4$) with no customer reactions. The ASE's are reported in units of 10^{-3}. We vary the number of arrivals, n, in our sample and the initial state of the system.

4.1. Description of the Experiments

In an effort to make our simulation study general and relevant, and to verify the robustness of our results, we explore the impact of several modelling assumptions on the magnitudes of the above errors. In particular, we consider the following:

1. Alternative system sizes, ranging from 50 to 1000; we generally focus on large systems where stochastic fluctuations are relatively small to reduce the effect of "noisy" observations.

2. Alternative forms for the abandonment-rate function, $\theta(\cdot)$. In particular we consider three functions: $\theta(w) = a + bw$, $\theta(w) = b - e^{-aw}$, and $\theta(w) = b + e^{aw}$, for different parameter values a and b. We vary the values of a and b so as to study the impact of customer response on the magnitudes of the reported queue-length and wait-time errors. Indeed, having a larger a value corresponds to "stronger" customer reactions. We vary a and b in such a way to keep the expected waiting time in the system constant; we do so in order to maintain comparable levels of system congestion (more precisely, we keep the fluid approximation of the waiting time constant)

3. Alternative values for the traffic intensity so as to vary the level of congestion in the system. In particular, we consider $\rho = 1.2$ and $\rho = 1.4$.

4. Both time-varying and stationary arrivals. For time-varying arrivals, we consider the function form in equation (35) of the main paper and vary both the frequency and

N	ASE(LES)	Waiting time	RASE(LES)	$N\times$ASE(LES)
10	0.0911	0.251	1.20	0.911
	$\pm 7.4\times 10^{-5}$	3.3×10^{-4}	1.4×10^{-4}	
30	1.66×10^{-2}	0.140	0.921	0.497
	$\pm 2.4\times 10^{-5}$	$\pm 1.0\times 10^{-4}$	$\pm 8.3\times 10^{-5}$	
50	7.54×10^{-3}	0.110	0.791	0.377
	$\pm 9.1\times 10^{-6}$	$\pm 2.1\times 10^{-4}$	$\pm 1.3\times 10^{-4}$	
70	4.58×10^{-3}	9.41×10^{-2}	0.719	0.320
	$\pm 5.6\times 10^{-6}$	$\pm 1.5\times 10^{-4}$	$\pm 1.5\times 10^{-4}$	
100	2.64×10^{-3}	7.82×10^{-2}	0.658	0.264
	$\pm 3.6\times 10^{-6}$	$\pm 1.9\times 10^{-4}$	$\pm 1.8\times 10^{-4}$	
300	5.07×10^{-4}	4.58×10^{-2}	0.491	0.152
	$\pm 5.6\times 10^{-6}$	$\pm 7.1\times 10^{-4}$	$\pm 7.0\times 10^{-4}$	
500	2.36×10^{-4}	3.54×10^{-2}	0.434	0.118
	$\pm 6.5\times 10^{-6}$	$\pm 8.3\times 10^{-4}$	8.8×10^{-4}	
700	1.42×10^{-4}	3.01×10^{-2}	0.396	0.0995
	$\pm 3.0\times 10^{-6}$	$\pm 9.2\times 10^{-4}$	$\pm 9.2\times 10^{-4}$	
1000	8.39×10^{-5}	2.56×10^{-2}	0.358	0.0839
	$\pm 2.3\times 10^{-6}$	$\pm 9.4\times 10^{-4}$	$\pm 1.1\times 10^{-3}$	

Table 3: Performance of the LES announcement in the $M/M/N+M$ model in the QED regime ($\rho = 1.0$) in steady state. We vary the number of servers, N. We report point and 95% confidence interval estimates of ASE(LES) and waiting times in the system. We also compute point estimates of RASE(LES).

amplitude of the sinusoidal arrival-rate function.

In the above simulation experiments, and for each simulated model, we collect the relative queue-length errors reported, and partition these into the following intervals:

$$(0, 0.05), (0.05, 0.1), (0.1, 0.2), (0.2, 0.3), (0.3, 0.4), (0.4, 0.5), \text{ and } (0.5, 1).$$

For example, the first interval corresponds to queue-length errors that are smaller than 5%, while the second interval corresponds to queue-length errors which are between 5% and 10%. For each interval, we collect the corresponding relative wait-time errors in the simulation run. For example, we collect all relative wait-time errors which correspond to queue-length errors that are smaller than 5% (first interval), or those which correspond to queue-length errors that are between 5% and 10% (second interval), and so on. As such, we collect data about wait-time errors corresponding to each of the queue-length error intervals above. We then compute summary statistics (on the wait-time errors) to assess precisely how the error in the queue length translates into the wait-time error. These are the $\bar{W}$ columns in the tables.

N	ASE(LES)	Waiting time	RASE(LES)	$N\times$ASE(LES)
10	0.143	0.673	0.562	1.43
	$\pm 2.1 \times 10^{-4}$	$\pm 6.1 \times 10^{-4}$	$\pm 7.7 \times 10^{-5}$	
30	4.01×10^{-2}	0.627	0.320	1.20
	$\pm 2.0 \times 10^{-4}$	$\pm 1.5 \times 10^{-3}$	$\pm 9.7 \times 10^{-5}$	
50	2.33×10^{-2}	0.618	0.247	1.17
	$\pm 1.3 \times 10^{-4}$	$\pm 1.3 \times 10^{-3}$	$\pm 7.7 \times 10^{-5}$	
70	1.64×10^{-2}	0.614	0.209	1.15
	$\pm 9.8 \times 10^{-5}$	$\pm 1.1 \times 10^{-3}$	$\pm 7.1 \times 10^{-5}$	
100	1.14×10^{-2}	0.612	0.175	1.14
	$\pm 7.1 \times 10^{-5}$	$\pm 1.1 \times 10^{-3}$	$\pm 6.4 \times 10^{-5}$	
300	3.75×10^{-3}	0.609	0.101	1.13
	$\pm 4.3 \times 10^{-5}$	± 1.05	$\pm 7.0 \times 10^{-5}$	
500	2.26×10^{-3}	0.609	7.81×10^{-2}	1.13
	$\pm 3.4 \times 10^{-5}$	$\pm 1.9 \times 10^{-3}$	$\pm 5.3 \times 10^{-5}$	
700	1.61×10^{-3}	0.607	6.60×10^{-2}	1.13
	$\pm 1.3 \times 10^{-5}$	$\pm 1.3 \times 10^{-3}$	$\pm 3.2 \times 10^{-5}$	
1000	1.11×10^{-3}	0.607	5.49×10^{-2}	1.11
	$\pm 1.3 \times 10^{-5}$	$\pm 1.7 \times 10^{-3}$	$\pm 3.6 \times 10^{-5}$	

Table 4: Performance of the LES announcement in the $M/M/N+M$ model in the ED regime ($\rho = 1.4$) in steady state. We vary the number of servers, N. We report point and 95% confidence interval estimates of ASE(LES) and waiting times in the system. We also compute point estimates of RASE(LES).

Additionally, for each queue-length interval, we report the sample size of such queue-length errors; these are the n columns in the tables.

4.2. Remarks

Based on our simulation results, we can formulate the following remarks on how the relative queue-length and wait-time errors are related in the system. In general, we also observe that Tables 5-10 show that wait-time errors typically fluctuate less extremely than queue-length errors.

- *Impact of N.* As can be seen from Tables 5-7, increasing the number of servers leads to decreasing the magnitude of relative wait-time errors in the system. That is not surprising due to the economies of scale in the system. For example, for $N = 50$, the median waiting time value corresponding to queue-length errors that are smaller than 5% is as high as roughly 20%. However, this error is close to 4% for $N = 1000$.
- *Impact of ρ.* As can be seen in comparing Tables 7 and 8, the impact of varying ρ is not

entirely straightforward. Indeed, we expect that increasing ρ would lead to a rise in the waiting times in the system. However, we observe that as ρ increases, the median wait-time value corresponding to small queue-length errors (e.g., < 0.05) decreases, whereas the median wait-time value corresponding to large queue-length errors (> 0.5) increases.

- *Impact of time-varying arrivals.* From Table 10, we see that the impact of varying the frequency of the arrival-rate function, γ is complicated as well. For example, comparing columns 1 and 3 of the table shows that increasing γ leads to increasing the median wait-time value for small values of queue-length errors, but it leads to decreasing the median wait-time value for large values of queue-length errors. On the other hand, increasing the amplitude of the arrival-rate function, α, leads to increasing the median wait-time error value throughout.

- *Impact of the abandonment-rate function.* In comparing our results across Tables 5-10, we see that the abandonment-rate function does not have a strong impact and the results we obtain for all functions considered are comparable.

Table 5: $N = 50, \rho = 1.4$

	$\theta(w) = b - \exp(-aw)$									
	$a=0, b=2$		$a=0.5, b=1.85$		$a=1, b=1.71$		$a=1.5, b=1.6$		$a=2, b=1.51$	
QL	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n
< 0.05	0.199	3990	0.186	4137	0.201	3980	0.200	4157	0.190	4132
$\in (0.05, 0.1)$	0.207	3937	0.193	3916	0.200	3785	0.206	3928	0.194	3883
$\in (0.1, 0.2)$	0.203	7023	0.204	7071	0.202	6956	0.205	7062	0.201	7075
$\in (0.2, 0.3)$	0.214	5573	0.215	5628	0.218	5687	0.215	5508	0.200	5645
$\in (0.3, 0.4)$	0.234	4234	0.230	4330	0.234	4275	0.233	4216	0.235	4450
$\in (0.4, 0.5)$	0.257	3081	0.259	2966	0.259	2930	0.250	2982	0.261	2991
> 0.5	0.315	6472	0.313	6228	0.314	6266	0.316	6399	0.299	6034
	$\theta(w) = aw + b$									
	$a=0, b=1$		$a=0.5, b=0.83$		$a=1, b=0.66$		$a=1.5, b=0.5$		$a=2, b=0.33$	
QL	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n
< 0.05	0.199	3990	0.186	4135	0.204	4274	0.198	3875	0.213	3794
$\in (0.05, 0.1)$	0.207	3937	0.196	3939	0.209	3981	0.199	3844	0.207	3682
$\in (0.1, 0.2)$	0.203	7023	0.203	7060	0.213	7238	0.205	7377	0.219	6953
$\in (0.2, 0.3)$	0.214	5573	0.215	5571	0.215	5574	0.223	6039	0.232	5516
$\in (0.3, 0.4)$	0.234	4234	0.229	4346	0.231	4335	0.240	4437	0.242	4577
$\in (0.4, 0.5)$	0.257	3081	0.252	2982	0.244	2921	0.256	2996	0.259	3177
> 0.5	0.315	6472	0.312	6243	0.298	6078	0.286	6041	0.293	6557
	$\theta(w) = b + \exp(aw)$									
	$a=0, b=0$		$a=0.5, b=-0.18$		$a=1, b=-0.4$		$a=1.5, b=-0.66$		$a=2, b=-0.96$	
QL	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n
< 0.05	0.199	3990	0.203	4004	0.201	4123	0.218	4090	0.225	3665
$\in (0.05, 0.1)$	0.207	3937	0.207	3856	0.198	3980	0.209	3993	0.233	3843
$\in (0.1, 0.2)$	0.203	7023	0.216	6968	0.205	7190	0.222	7136	0.236	6944
$\in (0.2, 0.3)$	0.214	5573	0.227	5491	0.213	5571	0.227	5447	0.248	5521
$\in (0.3, 0.4)$	0.234	4234	0.239	4167	0.227	4452	0.238	4284	0.268	4462
$\in (0.4, 0.5)$	0.257	3081	0.254	2938	0.243	3102	0.261	2960	0.284	3164
> 0.5	0.315	6472	0.311	6608	0.311	6039	0.296	6253	0.325	6870

Table 6: $N = 100, \rho = 1.4$

$\theta(w) = b - \exp(-aw)$										
	$a=0, b=2$		$a=0.5, b=1.85$		$a=1, b=1.71$		$a=1.5, b=1.6$		$a=2, b=1.51$	
QL	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n
< 0.05	0.136	12363	0.137	11805	0.137	11693	0.138	11747	0.140	11989
$\in (0.05, 0.1)$	0.135	11577	0.134	11577	0.136	11667	0.136	11539	0.140	11405
$\in (0.1, 0.2)$	0.144	19256	0.145	19716	0.144	19441	0.143	19495	0.152	18738
$\in (0.2, 0.3)$	0.159	11985	0.155	12018	0.155	12348	0.155	12368	0.166	12229
$\in (0.3, 0.4)$	0.180	6796	0.178	6665	0.181	6632	0.180	6613	0.177	7120
$\in (0.4, 0.5)$	0.208	3322	0.201	3370	0.206	3407	0.211	3427	0.190	3876
> 0.5	0.253	3961	0.242	4004	0.235	3967	0.233	3966	0.218	4376

$\theta(w) = aw + b$										
	$a=0, b=1$		$a=0.5, b=0.83$		$a=1, b=0.66$		$a=1.5, b=0.5$		$a=2, b=0.33$	
QL	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n
< 0.05	0.136	12363	0.137	11683	0.139	11703	0.141	11913	0.142	11749
$\in (0.05, 0.1)$	0.135	11577	0.136	11593	0.138	11558	0.140	11117	0.147	10953
$\in (0.1, 0.2)$	0.144	19256	0.145	19689	0.144	19368	0.150	18985	0.155	18870
$\in (0.2, 0.3)$	0.159	11985	0.156	12105	0.159	12440	0.165	12450	0.168	12552
$\in (0.3, 0.4)$	0.180	6796	0.179	6691	0.181	6560	0.187	7269	0.182	7205
$\in (0.4, 0.5)$	0.208	3322	0.200	3386	0.206	3483	0.213	3729	0.194	3581
> 0.5	0.253	3961	0.239	4008	0.233	4043	0.239	4059	0.227	4448

$\theta(w) = b + \exp(aw)$										
	$a=0, b=0$		$a=0.5, b=-0.18$		$a=1, b=-0.4$		$a=1.5, b=-0.66$		$a=2, b=-0.96$	
QL	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n
< 0.05	0.136	12363	0.139	11613	0.140	11648	0.147	11591	0.155	11347
$\in (0.05, 0.1)$	0.135	11577	0.135	11717	0.138	11603	0.147	10994	0.151	10819
$\in (0.1, 0.2)$	0.144	19256	0.145	19538	0.145	19287	0.153	18786	0.161	18485
$\in (0.2, 0.3)$	0.159	11985	0.155	12167	0.160	12329	0.172	12390	0.174	12448
$\in (0.3, 0.4)$	0.180	6796	0.179	6708	0.183	6698	0.187	7536	0.188	7563
$\in (0.4, 0.5)$	0.208	3322	0.204	3410	0.209	3537	0.217	3908	0.218	3942
> 0.5	0.253	3961	0.237	4002	0.234	4053	0.244	4350	0.235	4754

Table 7: $N = 1000$, $\rho = 1.4$

$\theta(w) = b - \exp(-aw)$										
	$a = 0, b = 2$		$a = 0.5, b = 1.85$		$a = 1, b = 1.71$		$a = 1.5, b = 1.6$		$a = 2, b = 1.51$	
QL	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n
< 0.05	0.043	369153	0.043	367805	0.044	366215	0.044	365055	0.044	364351
$\in (0.05, 0.1)$	0.048	222987	0.049	222343	0.049	222811	0.049	222390	0.050	222663
$\in (0.1, 0.2)$	0.062	100609	0.062	102383	0.063	103489	0.063	104964	0.063	105274
$\in (0.2, 0.3)$	0.105	4179	0.108	4372	0.109	4362	0.108	4420	0.108	4526
$\in (0.3, 0.4)$	0.176	179	0.169	205	0.175	231	0.167	268	0.165	264
$\in (0.4, 0.5)$	0.185	62	0.210	62	0.202	62	0.237	75	0.231	91
> 0.5	0.405	227	0.407	226	0.407	226	0.414	224	0.421	227

$\theta(w) = aw + b$										
	$a = 0, b = 1$		$a = 0.5, b = 0.83$		$a = 1, b = 0.66$		$a = 1.5, b = 0.5$		$a = 2, b = 0.33$	
QL	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n
< 0.05	0.043	369153	0.043	367111	0.044	364401	0.044	361064	0.045	358869
$\in (0.05, 0.1)$	0.048	222987	0.049	222769	0.050	222611	0.050	223198	0.051	222028
$\in (0.1, 0.2)$	0.062	100609	0.062	102668	0.063	105281	0.063	107770	0.064	110748
$\in (0.2, 0.3)$	0.105	4179	0.109	4354	0.108	4552	0.110	4784	0.113	5141
$\in (0.3, 0.4)$	0.176	179	0.167	205	0.168	254	0.168	246	0.173	250
$\in (0.4, 0.5)$	0.185	62	0.210	63	0.232	73	0.221	105	0.224	116
> 0.5	0.405	227	0.407	226	0.414	224	0.421	229	0.407	244

$\theta(w) = b + \exp(aw)$										
	$a = 0, b = 0$		$a = 0.5, b = -0.18$		$a = 1, b = -0.4$		$a = 1.5, b = -0.66$		$a = 2, b = -0.96$	
QL	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n
< 0.05	0.043	369153	0.043	366642	0.044	361616	0.046	355608	0.048	342724
$\in (0.05, 0.1)$	0.048	222987	0.049	222621	0.050	223243	0.052	222565	0.054	225116
$\in (0.1, 0.2)$	0.062	100609	0.062	103232	0.063	107103	0.066	113035	0.069	121743
$\in (0.2, 0.3)$	0.105	4179	0.109	4406	0.112	4869	0.116	5549	0.120	7011
$\in (0.3, 0.4)$	0.176	179	0.174	209	0.167	256	0.175	278	0.182	421
$\in (0.4, 0.5)$	0.185	62	0.215	60	0.235	82	0.223	113	0.220	114
> 0.5	0.405	227	0.407	226	0.421	227	0.407	248	0.406	267

Table 8: $N = 1000$, $\rho = 1.2$

$\theta(w) = b - \exp(-aw)$

	$a=0, b=2$		$a=0.5, b=1.91$		$a=1, b=1.83$		$a=1.5, b=1.76$		$a=2, b=1.69$	
QL	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n
< 0.05	0.061	229744	0.061	229677	0.062	229605	0.062	229209	0.062	229178
$\in (0.05, 0.1)$	0.065	174689	0.065	174084	0.065	173847	0.065	173873	0.065	173615
$\in (0.1, 0.2)$	0.076	161901	0.076	162291	0.076	162288	0.076	162522	0.076	162740
$\in (0.2, 0.3)$	0.110	27058	0.110	27372	0.109	27741	0.109	27860	0.109	27882
$\in (0.3, 0.4)$	0.148	3289	0.150	3266	0.153	3178	0.151	3200	0.149	3211
$\in (0.4, 0.5)$	0.189	468	0.189	476	0.185	496	0.191	497	0.189	520
> 0.5	0.222	99	0.248	82	0.223	93	0.243	87	0.195	102

$\theta(w) = aw + b$

	$a=0, b=1$		$a=0.5, b=0.91$		$a=1, b=0.82$		$a=1.5, b=0.73$		$a=2, b=0.64$	
QL	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n
< 0.05	0.061	229744	0.061	229584	0.062	229455	0.062	228769	0.062	227500
$\in (0.05, 0.1)$	0.065	174689	0.065	174141	0.065	173730	0.065	173868	0.065	174723
$\in (0.1, 0.2)$	0.076	161901	0.076	162320	0.076	162495	0.076	162762	0.077	162733
$\in (0.2, 0.3)$	0.110	27058	0.110	27396	0.109	27798	0.109	27991	0.110	28401
$\in (0.3, 0.4)$	0.148	3289	0.151	3251	0.152	3190	0.150	3214	0.151	3248
$\in (0.4, 0.5)$	0.189	468	0.190	472	0.186	488	0.192	545	0.190	538
> 0.5	0.222	99	0.244	84	0.225	92	0.213	99	0.191	105

$\theta(w) = b + \exp(aw)$

	$a=0, b=0$		$a=0.5, b=-0.1$		$a=1, b=-0.2$		$a=1.5, b=-0.31$		$a=2, b=-0.44$	
QL	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n
< 0.05	0.061	229744	0.061	229713	0.062	229083	0.062	227435	0.063	226054
$\in (0.05, 0.1)$	0.065	174689	0.065	173941	0.065	173717	0.065	174622	0.065	173919
$\in (0.1, 0.2)$	0.076	161901	0.076	162326	0.076	162821	0.077	162927	0.077	164029
$\in (0.2, 0.3)$	0.110	27058	0.109	27488	0.109	27847	0.109	28385	0.110	29311
$\in (0.3, 0.4)$	0.148	3289	0.153	3221	0.152	3192	0.150	3227	0.150	3284
$\in (0.4, 0.5)$	0.189	468	0.188	475	0.191	498	0.189	544	0.189	542
> 0.5	0.222	99	0.244	84	0.233	90	0.186	108	0.194	109

Table 9: $N = 1000$, $\rho = 1.4$

	$\theta(w) = b - \exp(-aw)$									
	$a = 1, b = 1.71$		$a = 1, b = 1.71$		$a = 1, b = 1.71$		$a = 1, b = 1.71$		$a = 1, b = 1.71$	
QL	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n
< 0.05	0.040	362607	0.043	271332	0.053	73663	0.038	379393	0.037	194097
$\in (0.05, 0.1)$	0.047	190956	0.055	190596	0.074	80470	0.043	186382	0.050	171128
$\in (0.1, 0.2)$	0.071	90816	0.088	149420	0.127	191426	0.062	70611	0.089	150210
$\in (0.2, 0.3)$	0.130	16441	0.138	34443	0.192	214549	0.138	6232	0.157	43574
$\in (0.3, 0.4)$	0.199	6455	0.194	12583	0.221	101173	0.190	1245	0.196	17163
$\in (0.4, 0.5)$	0.244	3326	0.247	5738	0.226	29165	0.228	612	0.249	5769
> 0.5	0.367	7886	0.345	11418	0.257	7467	0.340	1558	0.321	5198
	$\theta(w) = aw + b$									
	$a = 1, b = 0.82$		$a = 1, b = 0.82$		$a = 1, b = 0.82$		$a = 1, b = 0.82$		$a = 1, b = 0.82$	
QL	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n
< 0.05	0.041	353625	0.045	272148	0.054	73379	0.034	406024	0.036	191481
$\in (0.05, 0.1)$	0.047	199054	0.055	189176	0.073	80832	0.041	173289	0.053	153417
$\in (0.1, 0.2)$	0.067	95425	0.083	153882	0.127	189258	0.060	58277	0.091	165069
$\in (0.2, 0.3)$	0.119	16953	0.138	33664	0.192	213979	0.131	6064	0.144	49434
$\in (0.3, 0.4)$	0.191	6224	0.189	11102	0.220	101470	0.203	1431	0.197	15203
$\in (0.4, 0.5)$	0.234	3426	0.220	5476	0.226	30562	0.259	670	0.242	6518
> 0.5	0.388	7670	0.330	9300	0.258	8433	0.309	1467	0.320	5883
	$\theta(w) = b + \exp(aw)$									
	$a = 1, b = -0.4$		$a = 1, b = -0.4$		$a = 1, b = -0.4$		$a = 1, b = -0.4$		$a = 1, b = -0.4$	
QL	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n
< 0.05	0.043	345810	0.046	268335	0.054	73972	0.038	379393	0.037	194097
$\in (0.05, 0.1)$	0.049	193785	0.055	197577	0.074	78759	0.043	186382	0.050	171128
$\in (0.1, 0.2)$	0.066	105494	0.087	150441	0.127	189439	0.062	70611	0.089	150210
$\in (0.2, 0.3)$	0.129	15089	0.140	31623	0.190	212038	0.138	6232	0.157	43574
$\in (0.3, 0.4)$	0.190	6027	0.179	11460	0.218	101839	0.190	1245	0.196	17163
$\in (0.4, 0.5)$	0.252	2932	0.220	5309	0.227	32061	0.228	612	0.249	5769
> 0.5	0.376	7031	0.342	9508	0.259	9805	0.340	1558	0.321	5198

Table 10: $N = 1000$, $\rho = 1.4$, time-varying arrivals

$\theta(w) = b - \exp(-aw)$												
	$a=1, b=1.71,$ $\gamma=0.044, \alpha=0.3$		$a=1, b=1.71$ $\gamma=0.262, \alpha=0.3$		$a=1, b=1.71$ $\gamma=1.571, \alpha=0.3$		$a=1, b=1.71$ $\gamma=0.044, \alpha=0.7$		$a=1, b=1.71$ $\gamma=0.262, \alpha=0.7$		$a=1, b=1.71$ $\gamma=1.571, \alpha=0.7$	
QL	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n
< 0.05	0.040	362607	0.043	271332	0.053	73663	0.038	379393	0.037	194097	0.074	21820
$\in (0.05, 0.1)$	0.047	190956	0.055	190596	0.074	80470	0.043	186382	0.050	171128	0.104	44463
$\in (0.1, 0.2)$	0.071	90816	0.088	149420	0.127	191426	0.062	70611	0.089	150210	0.171	44392
$\in (0.2, 0.3)$	0.130	16441	0.138	34443	0.192	214549	0.138	6232	0.157	43574	0.220	49253
$\in (0.3, 0.4)$	0.199	6455	0.194	12583	0.221	101173	0.190	1245	0.196	17163	0.263	53097
$\in (0.4, 0.5)$	0.244	3326	0.247	5738	0.226	29165	0.228	612	0.249	5769	0.393	337775
> 0.5	0.367	7886	0.345	11418	0.257	7467	0.340	1558	0.321	5198	0	0

$\theta(w) = aw + b$												
	$a=1, b=0.82$ $\gamma=0.044, \alpha=0.3$		$a=1, b=0.82$ $\gamma=0.262, \alpha=0.3$		$a=1, b=0.82$ $\gamma=1.571, \alpha=0.3$		$a=1, b=0.82$ $\gamma=0.044, \alpha=0.7$		$a=1, b=0.82$ $\gamma=0.262, \alpha=0.7$		$a=1, b=0.82$ $\gamma=1.571, \alpha=0.7$	
QL	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n
< 0.05	0.041	353625	0.045	272148	0.054	73379	0.034	406024	0.036	191481	0.055	22619
$\in (0.05, 0.1)$	0.047	199054	0.055	189176	0.073	80832	0.041	173289	0.053	153417	0.101	44727
$\in (0.1, 0.2)$	0.067	95425	0.083	153882	0.127	189258	0.060	58277	0.091	165069	0.165	48496
$\in (0.2, 0.3)$	0.119	16953	0.138	33664	0.192	213979	0.131	6064	0.144	49434	0.219	51021
$\in (0.3, 0.4)$	0.191	6224	0.189	11102	0.220	101470	0.203	1431	0.197	15203	0.271	55513
$\in (0.4, 0.5)$	0.234	3426	0.220	5476	0.226	30562	0.259	670	0.242	6518	0.381	336013
> 0.5	0.388	7670	0.330	9300	0.258	8433	0.309	1467	0.320	5883	0	0

$\theta(w) = b + \exp(aw)$												
	$a=1, b=-0.4$ $\gamma=0.044, \alpha=0.3$		$a=1, b=-0.4$ $\gamma=0.262, \alpha=0.3$		$a=1, b=-0.4$ $\gamma=1.571, \alpha=0.3$		$a=1, b=-0.4$ $\gamma=0.044, \alpha=0.7$		$a=1, b=-0.4$ $\gamma=0.262, \alpha=0.7$		$a=1, b=-0.4$ $\gamma=1.571, \alpha=0.7$	
QL	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n	$\bar{W}$	n
< 0.05	0.043	345810	0.046	268335	0.054	73972	0.038	379393	0.037	194097	0.074	21820
$\in (0.05, 0.1)$	0.049	193785	0.055	197577	0.074	78759	0.043	186382	0.050	171128	0.104	44463
$\in (0.1, 0.2)$	0.066	105494	0.087	150441	0.127	189439	0.062	70611	0.089	150210	0.171	44392
$\in (0.2, 0.3)$	0.129	15089	0.140	31623	0.190	212038	0.138	6232	0.157	43574	0.220	49253
$\in (0.3, 0.4)$	0.190	6027	0.179	11460	0.218	101839	0.190	1245	0.196	17163	0.263	53097
$\in (0.4, 0.5)$	0.252	2932	0.220	5309	0.227	32061	0.228	612	0.249	5769	0.393	337775
> 0.5	0.376	7031	0.342	9508	0.259	9805	0.340	1558	0.321	5198	0.393	337775